Metodi di analisi di superfici discrete e loro applicazioni

Silvia Biasotti

Giuseppe Patané

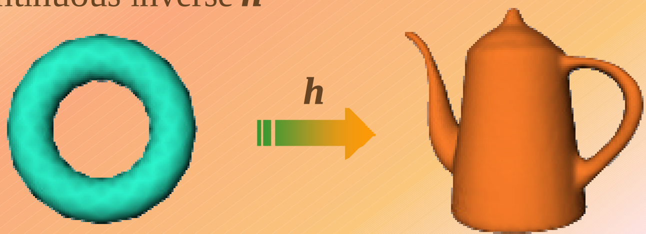
GE-IMATI-CNR

outline

- tools for analysis and synthesis
  - elements of geometry and topology
    - concepts of
      - homeomorphism
      - manifold and surface
      - critical point
      - curvature
      - genus
      - homotopy and homology
  - Morse theory
  - Reeb graphs

homeomorphism

- a homeomorphism between to topological spaces X and Y is a continuous bijection ***h***:X→Y with continuous inverse $\boldsymbol{h}^{-1}$



- differential topology deals with the relations among topology and critical points of a function defined on a smooth manifold

basic concepts

- (Smooth function): *Let X an arbitrary subset of Â*^{n}*. Then a function* ***f*** *:X® Â*^{m} *is called smooth if for every point x Î X there is an open set U Í Â*^{n} *and a function* ***F****: U® Â*^{m} *such that F=f*$_{|X}$ *on XÇU and F has continuous partial derivatives of all orders.*
- *Given X Í Â*^{n} *and Y Í Â*^{m}*; if the smooth function f:X® Y is bijective and* $\boldsymbol{f}^{-1}$ *is also smooth, the function f is a diffeomorphism*

basic concepts

- (manifold without boundary) *A topological Haussdorff space **M** is called a k-dimensional topological manifold if each point m Î M admits a neighborhood* U_i *Í **M** homeomorphic to the open disk* D^k*={x Î* $Â^k$ *| |x|<1} and **M**=*$È_{i Î N}\ U_i$
- (manifold with boundary) *A topological Haussdorff space **S** is called a k-dimensional topological manifold with boundary if each point m Î **M** admits a neighborhood* U_i *Í **M** homeomorphic either to the open disk* D^k*={x Î* $Â^k$*| |x|<1} or the open half-space Âx{y Î Â | y ³ 0} and **M**=*$È_{i Î N}\ U_i$
- ***k*** is called the *dimension* of the manifold

basic concepts

- (transition functions) *Let {*(U_i, j_i)*} an union of charts* $j_i : U_i ® D^k$ *on a k-dimensional manifold **M**. The homeomorphisms* $j_{i,j} : j_i(U_i \cap U_j) \rightarrow j_j(U_i \cap U_j)$ *such that* $j_{i,j} = j_j \cap j_i^{-1}$ *are called* **transition functions**
- (smooth manifold) *A k-dimensional topological manifold with (resp. without) boundary is called a* **smooth manifold** *with (resp. without) boundary, if all transition functions* $j_{i,j}$ *are smooth*
- (orientability) *A manifold **M** is called* **orientable** *is there exists an atlas {*(U_i, j_i)*} on it such that the Jacobian of all transition functions is positive for all intersecting pairs of regions*

surfaces

- (surface without boundary) *A topological Haussdorff space* $\mathbf{S}$ *is called a surface if each point* $s \in S$ *admits a neighborhood* $U_i \subseteq \mathbf{S}$ *homeomorphic to the open disk* $D^2=\{x \in \mathbb{R}^2 \mid |x|<1\}$ *and* $\mathbf{S}=\cup_{i \in N} U_i$
- (surface with boundary) *A topological Haussdorff space* $\mathbf{S}$ *is called a surface if each point* $s \in \mathbf{S}$ *admits a neighborhood* $U_i \subseteq \mathbf{S}$ *homeomorphic either to the open disk* $D^2=\{x \in \mathbb{R}^2 \mid |x|<1\}$ *or the open half-space* $\mathbb{R} x \{y \in \mathbb{R} \mid y \geq 0\}$ *and* $\mathbf{S}=\cup_{i \in N} U_i$

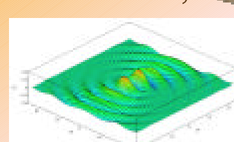
examples

- 3-manifolds with boundary: a solid sphere, a solid torus, a solid knot



- 2-manifolds: a sphere, a torus

- 2-manifold with boundary: a sphere with 2 holes, single-valued functions *(scalar fields)*



- 1 manifold: a circle, a line

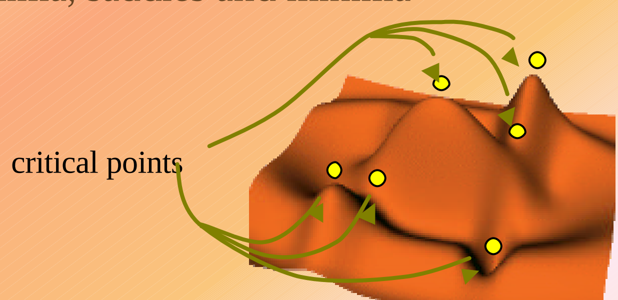
parametric surfaces

- regular parameterisation of a surface:
  $F : U \subset R^2 \rightarrow R^3$
  such that
  - $F(u,v) \in S$, for all $(u,v) \in U$
  - $F(u,v) = (x(u,v), y(u,v), z(u,v))$, such that

$$\begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix} \times \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

critical points

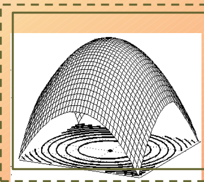
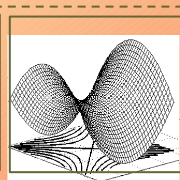
- the perception of shape is focused where the topological changes occur, for example at maxima, saddles and minima



critical points

- given a smooth function f defined on a manifold
  - a point x is called ***regular*** if the differential df_x is an epimorphism
  - a point x is called ***critical*** is the differential df_x is the zero map
  - a critical point x is called ***non-degenerate*** if the Hessian matrix H is the second partial derivatives of f is non-singular at that point
  - if x is a non-degenerate critical point of f, the number λ of negative eigenvalues of H is called the ***index*** of x

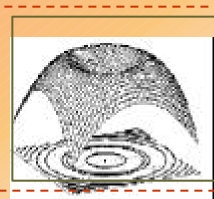
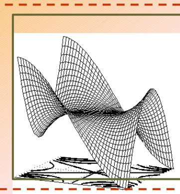
critical points on a surface

non-degenerate c. p.

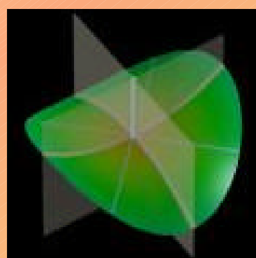
$\lambda=2$ $\lambda=1$

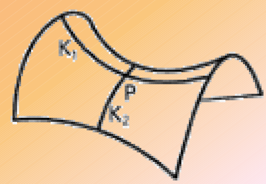
degenerate c. p.

principal curvatures

- the ***principal curvatures*** measure the maximum and minimum bending of a surface at each point

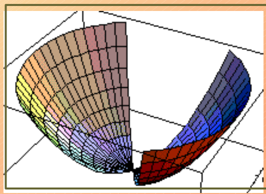




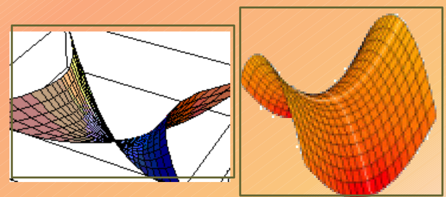
Gaussian and mean curvature

- given k_1 and k_2 the principal curvatures at a point surface
  - gaussian curvature $K=k_1k_2$
  - mean curvature $H=(k_1+k_2)/2$
- the gaussian curvature is an intrinsic property of the surface, that is invariant with respect to isometries
- according to behavior of the sign of K, the points of a surface may be classified as
  - elliptic
  - hyperbolic
  - parabolic or planar

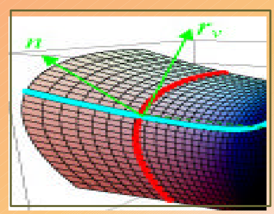
examples



K>0



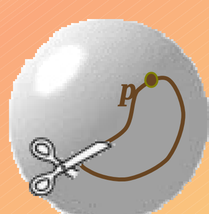
K<0

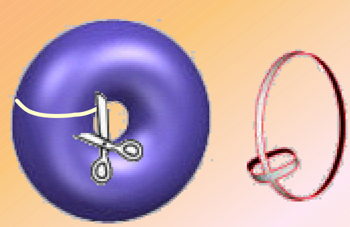


K=0, H≠0

loops on a surface

- a *loop* is a closed curve whose initial and final points coincide in a fixed point *p* known as the basepoint





homotopy

- an homotopy is a continuous transformation from one function to another
- the fundamental group of an arcwise-connected set X is the group formed by the sets of equivalence classes of the set of all loops under the equivalence relation of homotopy.
- the fundamental group only depends on the homotopy type of X *(Massey '67, Munkres '00)*
- homotopy groups generalize the fundamental group to higher dimensions

homology

- the homology of a space reflects its topology and uses finite generated abelian groups to analyse that
- singular homology groups form a measure of the hole structure of a space
- singular homology allows the definition of homology groups for all cell complexes
- homology groups are an abelianization of the homotopy groups

homology

- the n^{th} homology group is the class made up of the n-cycles that bound regions all by themselves
  - the 0-th group measures the number of connected components;
  - the 1-st: the number of through holes;
  - the 2-nd: the number of cavities
  - etc..
- the most important homology group for smooth surfaces is the first homology group
- the most interesting homology groups for 3-manifolds are the first and the second one
- the i-th ***Betti number*** is the dimension of the i-th homology group

genus

- the genus *g* of a surface *S* without boundary is:
  - half the first Betti number of *S*
  - the cardinality of a minimal set of mutually non-isotopic loops with the property that is a connected planar surface
- any orientable surface without boundary is a connected sum of *g* tori, where *g* is its genus, *g*³*0*
- the genus of a surface with boundary is the genus of the surface *S'* obtained by gluing a disc onto each boundary component *(Massey '67)*
- the genus of a surface is topologically invariant

Euler formula

- *#maxima - #saddles + #minima= c(S)*

  (differential geometry)

- *2p (v - e + f) = K(S)*

  (differential topology)

- *v - e + f = c(S) =2-2g*

  (algebraic topology)

Morse theory

- Morse theory studies the relationship between a function's critical points and the topology of its domain
- it indicates when the topological type changes and what kind of changes take place
- it provides a surface decomposition into a limited set of primitive topological cells, defined by the surface critical points and their corresponding index
- a function *f* is called Morse if all of its critical points are non-degenerate

does any Morse function exist?

- on any smooth compact manifold there exist Morse functions
- Morse functions are everywhere dense in the space of all smooth functions on the manifold
- any Morse function has only a finite number of critical points on a compact manifold
- the set S of all simple Morse functions is everywhere dense in the set of all Morse functions
- examples of Morse functions on a smooth manifold: *height function*, *distance functions*, *geodesic distance*, *etc*.

Morse theory & critical points

- Weak Morse inequalities
  - C_i=#{critical points of index i} and b_i i-th Betti number of *M* then
  - b_i £ C_i
  - $S(-1)^i C_i = S(-1)^i b_i = c(M)$
- Strong Morse inequalities
  - "i ³0, $b_i - b_{i-1} + - \ldots \pm b_0$ £ $C_i - C_{i-1} + - \ldots \pm C_0$

#maxima - #saddles + #minima = *c* (M) = 2-2g

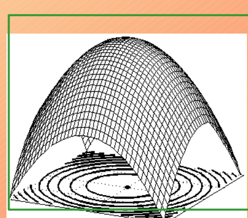
Morse theory & critical point configuration

- (Morse Lemma)

  in a neighbourhood of each non-degenerate critical point *P*, the function *f* can be expressed as:

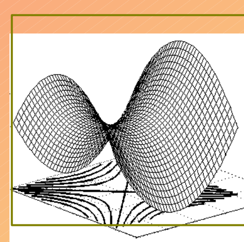
$$f = f(P) - (y_1)^2 - \ldots - (y_l)^2 + (y_{l+1})^2 + \ldots + (y_n)^2$$

2-manifolds



$f = -x^2 - y^2$

maximum
l =2

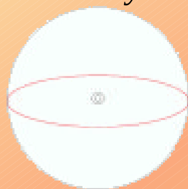


$f = -x^2 + y^2$

saddle
l =1

3-manifolds

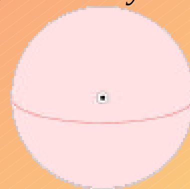
$f = +x^2 + y^2 + z^2$



minimum

$\lambda = 0$

$f = -x^2 - y^2 - z^2$



maximum

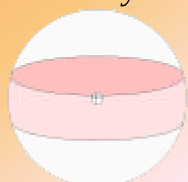
$\lambda = 3$

$f = -x^2 + y^2 + z^2$

saddle

$\lambda = 1$

$f = -x^2 - y^2 + z^2$

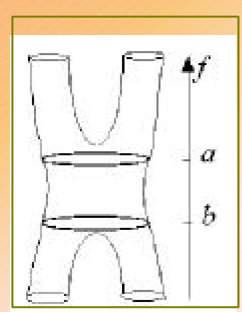


saddle

$\lambda = 2$

Morse theory & critical points

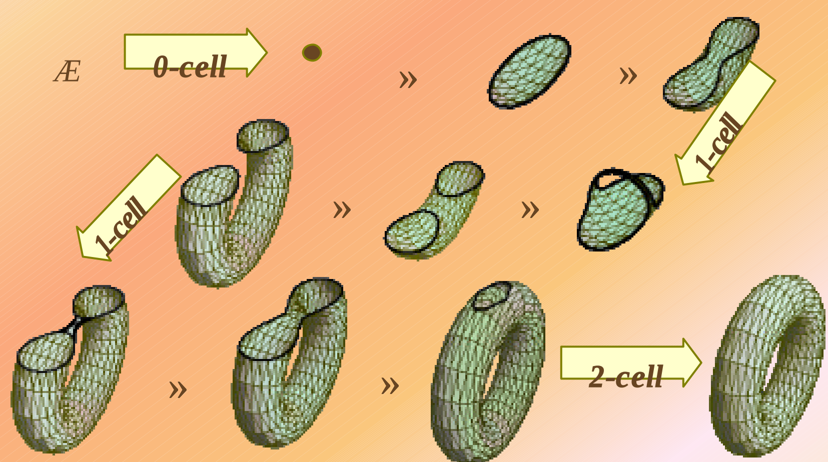
- let $f: M \to R$ be a real valued function and let $[a, b] \subset R$ be an interval non containing critical values of f. The level sets $f^{-1}(a)$ and $f^{-1}(b)$ are *diffeomorphic*
- denoting $M^x = \{ p \in M \mid f(p) \leq x \}$ let P be a critical point such that $f(p)=c$. $\forall \varepsilon > 0$ such that $f^{-1}[c-\varepsilon, c+\varepsilon]$ contains other critical points of f, the set has the homotopy type of $M^{c-\varepsilon}$ cell attached



Morse theory & shape decomposition

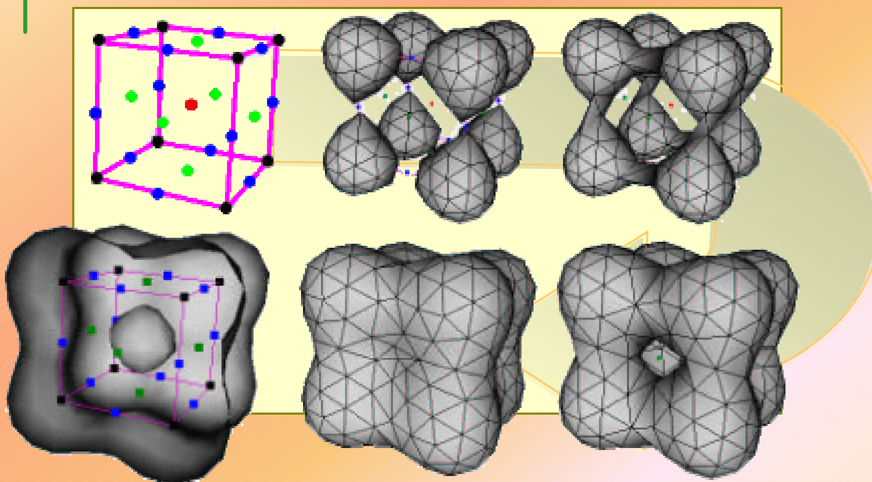
- Theorem *(CW complex decomposition)*
  - let *S* be a smooth compact manifold embedded in an Euclidean space. Let *f*: *S*®*R* be a smooth, real valued, Morse function on *S*. Then *S* is homeomorphic (i.e. topologically equivalent) to a cell complex of dimension ***i*** for each critical point of index ***i***

attaching cells: a torus



pictures from http://www.cs.rug.nl/~gert/topology.html

another example: a cuboid solid



Morse theory does not say

- that all smooth functions on S have the same number of critical points
- if the cell complex obtained using a given f is the "*best possible*" (i.e. it has the fewest number of cells)

some references

- V. Guillemin and A. Pollack, *Differential Topology*, Englewood Cliffs, NJ:Prentice Hall, 1974
- H. B. Griffiths, *Surfaces*, Cambridge University Press, 1976
- R. Engelking and K. Sielucki, *Topology: A geometric approach*, Sigma series in pure mathematics, Heldermann, Berlin, 1992
- A. Fomenko, *Visual Geometry and Topology*, Springer-Verlag, 1995
- W. Massey, *Algebraic topology: An Introduction*, Brace&World Inc., 1967
- A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2001
- J. Milnor, *Morse theory*, Princeton University Press, New Jersey, 1963

aim

- find a description of a surface shape which is:
  - meaningful — mathematically well-defined and related to *intuitive* shape features
  - high level — provide a structuring of shape features into a "morphological skeleton"
  - computable — affordable computational complexity

shape descriptors

- medial axis transform *(Blum 1967)*
- shock graphs *(Kimia,Tannenbaum, Zucker 1995)*
- surface networks *(Pfaltz 1976)*
- skeletons and centerlines *(Sethian 1985, Bloomenthal 1991)*
- apparent contours *(Haefliger 1960, Pignoni 1991)*
- size functions *(Ferri, Frosini 1990)*
- **Reeb graphs** *(Reeb 1946)*

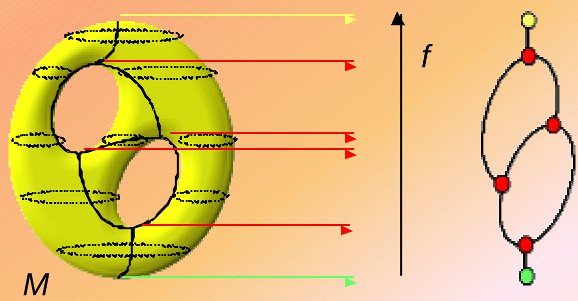
Reeb graph

- deals with tools of differential topology and is aimed to
  - describe a conceptual model for surface representation based on topology coding
  - define a sketch of the surface usable for shape abstraction
  - derive a topological descriptor of a geometric model which is able to describe the overall shape structure

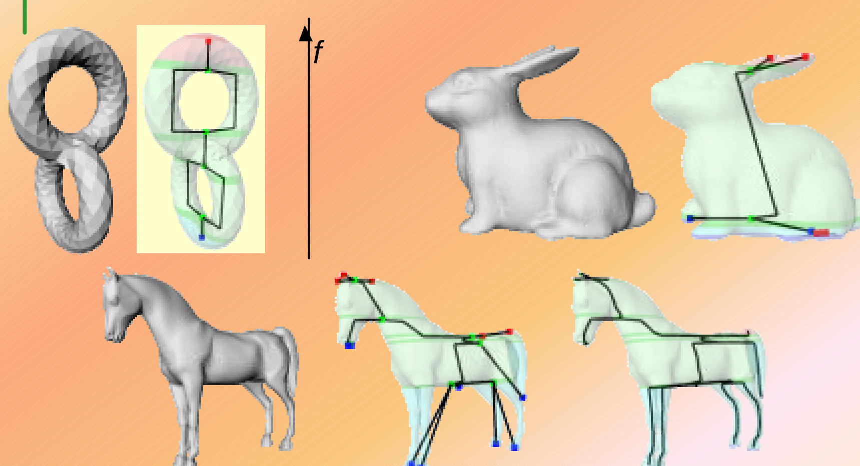
Reeb graph definition

given *f*: S→R defined on the surface S, the ***Reeb graph*** of S wrt *f* is the quotient space defined by "~" :

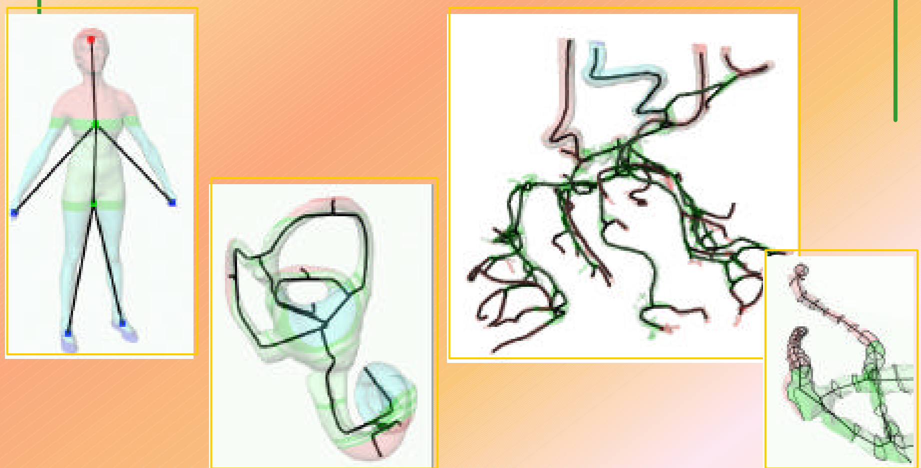
$(X_1, f(X_1)) \sim (X_2, f(X_2))$ Û $f(X_1) = f(X_2)$ and X_1 and X_2 are in the same connected component of $f^{-1}(f(X_1))$



Reeb graph: examples



More examples



Reeb graph properties

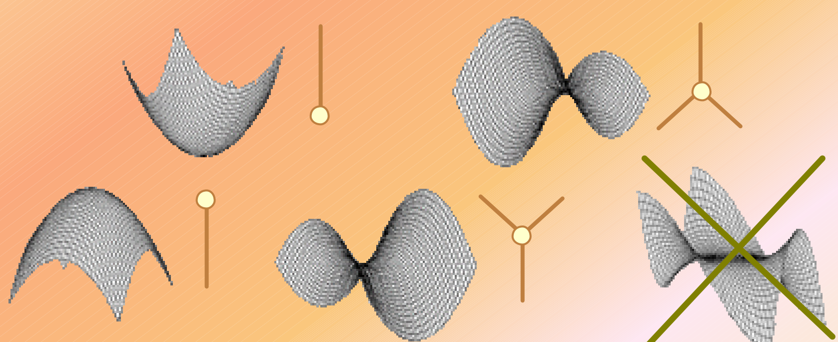
- it provides a 1D structure
- it roughly describes the shape of an object
- it is suited for multi-resolution
- the flexibility of the choice of the function *f* makes it adaptable to different tasks
  - simplification and compression
  - recognition
  - classification
  - shape retrieval
  - morphing

references

- G. Reeb, “Sur les point sunguilèrs d’une form del Pfaff complètement intégrable ou d’une fonction munèrique”, *Comptes Rendu de l’Academie des Sciences*, 222:847-849, 1946
- Y. Shinagawa, T. Kunii and Y. Kergosien, “Surface coding based on Morse Theory”, *IEEE Computer Graphics and Applications*, 11(5):66-78, 1991
- M. Hilaga, Y. Shinagawa, T. Komura and T. Kunii, “Topology matching for fully automatic similarity estimation of 3D shapes”, *Proc. SIGGRAPH 2001*, pp. 203-212, 2001
- K. Cole-McLaughlin, H. Edelsbrunner, J. Harer, V. Natarajan and V. Pascucci, “Loops in Reeb Graphs of 2-Manifolds ”, *Proc. of the 19th ACM Symposium on Computational Geometry*, 344-350, 2003

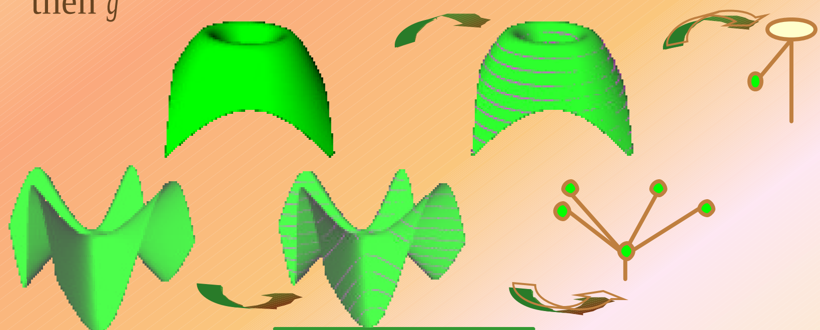
Reeb graph wrt Morse functions

- in case the function *f* is Morse, the configuration of Reeb nodes is simple



Reeb graph wrt continuous functions

- Reeb graphs are well defined also when the critical points of *f* are degenerate
- the number of loops of the graph may be greater then *g*



we are going to describe:

– Reeb Graph w.r.t height function

– Reeb Graph w.r.t distance from the barycentre

– Reeb Graph w.r.t geodesic distance

– Centerline based on discrete geodesic distance from a source point

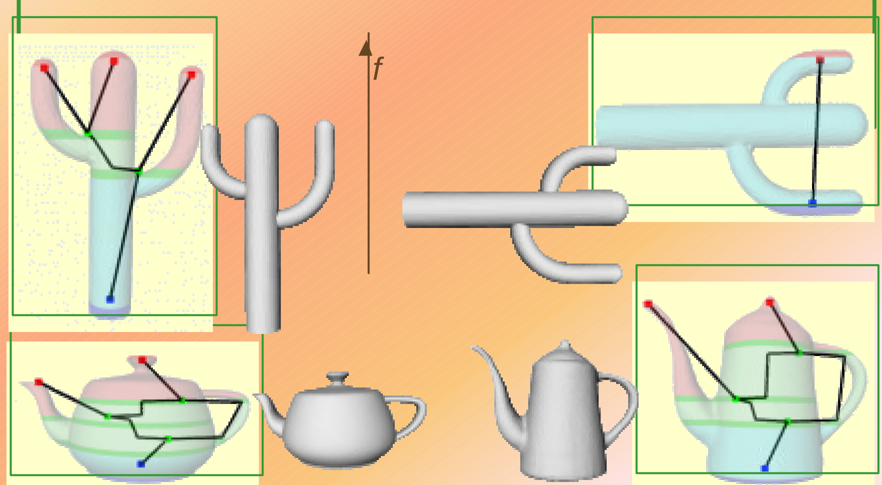
– Skeleton based on topological distance from curvature extrema

Reeb graph

- the Reeb graph of a given object changes according to the chosen mapping function *f*
- Problem: choose the “best” function, with an eye on
  - Computational cost
  - Invariance
  - Description effectiveness depending on the particular application

G.Reeb. Sur le points singuliers d'une forme de Pfaff completement integrable ou d'une fonctionn numerique. Comptes Rendus Acad. Science, Paris, 1946, 222: 847-849

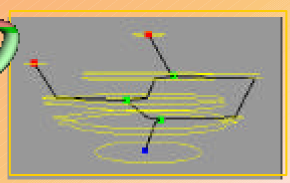
RG wrt the height function

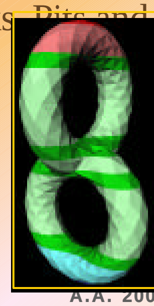


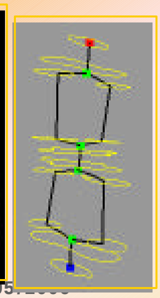
RG w.r.t. height function

– The equivalence classes introduced by the height function correspond to the intersection of the mesh with planes orthogonal to the z-axis direction.

– Critical points correspond to Peaks, Pits and Saddles.



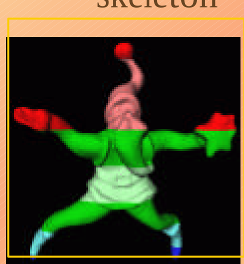


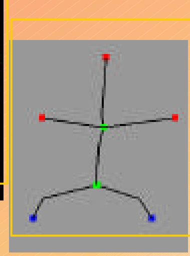


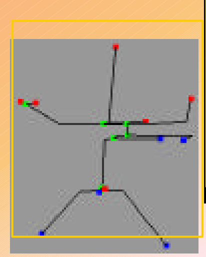
RG w.r.t. height function

– The function evaluation is immediate

– The corresponding graph intuitively reseambles the skeleton







Reeb Graph: height function

☺ Resemble the skeleton of the input shape

☺ An approximate shape can be extracted from the critical sections and the relations among them.

Reeb Graph: height function

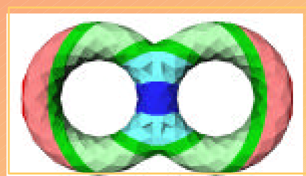
☺ Good computational cost of the graph construction $O((n+k)\log(n+k))$ (where k is the number of vertices inserted to the mesh during the slicing phase)

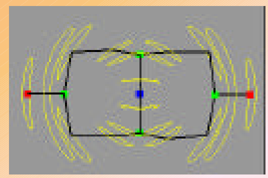
☹ It is dependent from rotations: the skeleton depends on the direction used.

☹ Not suitable for matching based applications.

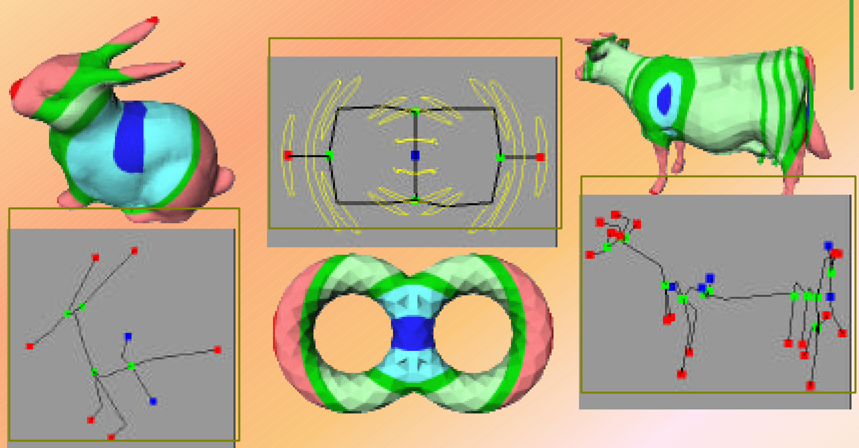
RG w.r.t. distance from the barycentre

– The equivalence classes correspond to the intersection of the mesh with a set of spheres centred in the barycentre.
– Maxima and minima correspond to protrusions and concavities wrt to the barycentre respectively.





RG wrt the distance from the center of mass



RG w.r.t. distance from the barycentre

- The barycentre is easy to compute and, due to its linear dependence on all the vertices, it is stable to small perturbations.
- The function is easy to compute.
- The corresponding graph is invariant to translation, rotation and uniform scaling of the object.

Reeb Graph: distance from barycentre

☺ Independent from object position in space

☺ It is suitable for matching purposes if the intent is to distinguish between different poses of the object (for instance a human body with the arms stretched or curled up).

Reeb Graph: distance from barycentre

☺ Good computational costs: $O(n)$ for the function evaluation, and $O((n+k)\log(n+k))$ for the graph extraction(where k is the number of vertices inserted to the mesh during the slicing phase)

☹ Not suitable for reconstruction purposes: isocontours are not planar.

RG w.r.t. geodesic distance

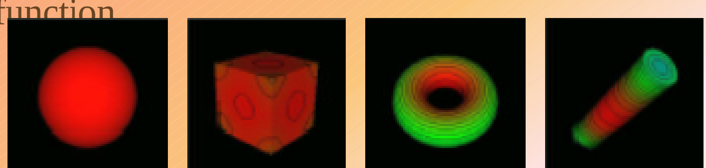
- The mapping function is

$$f(p)=\int_{v \text{ in } S} g(p,v)dS$$

where g represents the geodesic distance

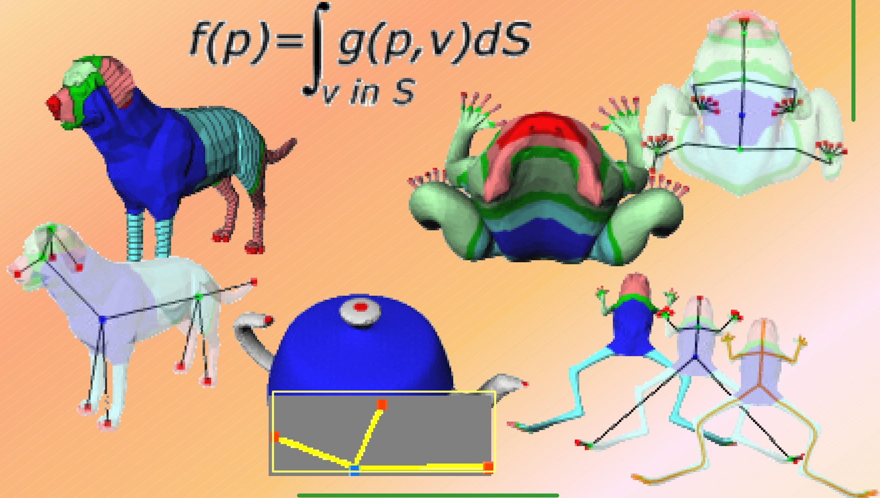
(M. Hilaga, Y. Shinagawa, T. Komura, T. L. Kunii, "Topology Matching for Fully Automatic Similarity Estimation of 3D Shapes", *Siggraph 2001*, 2001)

- Surface protrusions are maxima of the mapping function



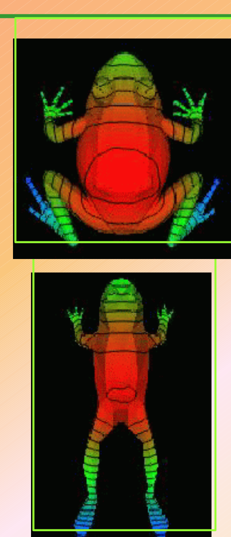
RG wrt the integral geodesic distance

$$f(p)=\int_{v \in S} g(p,v)\,dS$$



RG w.r.t. geodesic distance

- The measure is stable to object deformations, independent from the object position and from the posture (the classifications of the frog in 2 different postures are the same).
- *f* is computationally expensive to compute



Reeb Graph: geodesic distance

☺ Independent from object position in space

☺ It is suitable for matching purposes if the intent is not to distinguish among different poses of the same object.

Reeb Graph: Geodesic distance

☹ The computational cost of the function evaluation is $O(n^2 \log n)$ due to the Dijkstra's algorithm

☺ Good computational cost of the graph extraction is

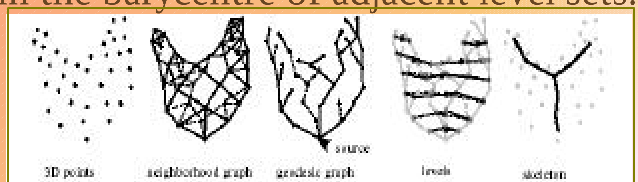
$$O(n+k)$$

(where k is the number of added vertices)

☹ Not suitable for reconstruction purposes: isocontours are not planar.

discrete geodesic distance from a seed point

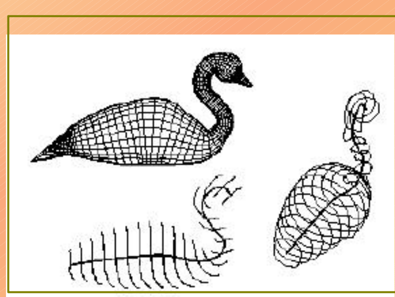
- choose a source point on the tip of an elongated feature
- compute level set of vertices w.r.t. using Dijkstra's algorithm
- join the barycentre of adjacent level sets.

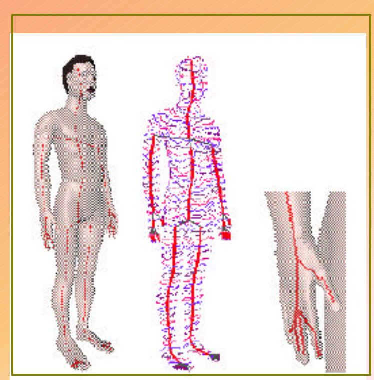


(A. Verroust and F. Lazarus. Extracting Skeletal Curves from 3D Scattered Data. In *Shape Modeling International '99*. Aizu, Japan, March 1999)

centerline based on discrete geodesic distance from a source point

- examples

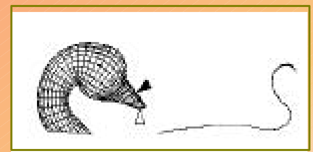


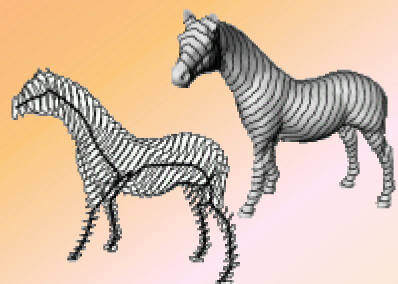


F. Lazarus and A. Verroust. Level Set Diagrams of Polyhedral Objects. In *Solid Modeling and Applications 1999*. Ann Arbour, June 1999

centerline based on discrete geodesic distance from a source point

- this method is effective on tubular shaped objects, where different source points don't compromise the resulting centerline.
- in other cases, the choise of a source point determines a priviledged slicing direction and eventually the the loss of some features (like the horse's ears).

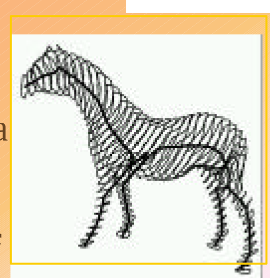


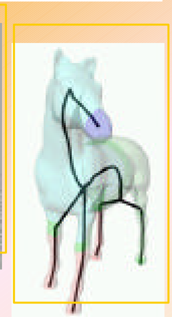


F. Lazarus and A. Verroust. Level Set Diagrams of Polyhedral Objects. In *Solid Modeling and Applications 1999.* Ann Arbour, June 1999

Discrete geodesic distance from a seed point

- This method is effective on tubular shaped objects, where different source points don't compromise the resulting centerline.
- In other cases, the choise of a source point determines a priviledged slicing direction and eventually the the loss of some features (like the horse's ears).





Centerline extraction

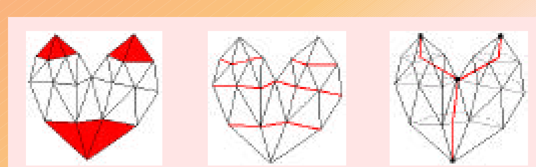
☺ It is intuitive and useful for finding centerline of tubular shapes

☺ Good computational cost: $O(n \log n)$

☹ Is not easy find out a a starting point that yields an expressive skeleton if the object is not tubular.

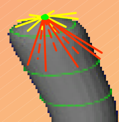
Skeleton based on topological distance from curvature extrema

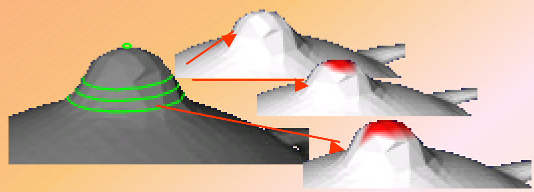
- Similar to the centreline extraction, but uses more source points
- Seed points are maxima of curvature
- Level sets are obtained by topological expansion of geodesic circles from curvature extrema



skeleton from curvature extrema

- seed points are maxima of curvature
- level sets are obtained by topological expansion of geodesic circles from curvature extrema
- curvature extrema are obtained by using a deficit angle measure computed on a neighbourhood of variable size

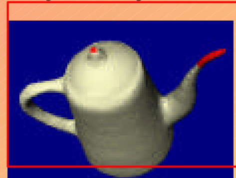


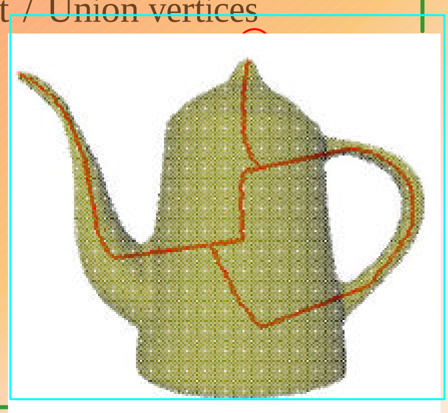


(Mortara, Patané, Spagnuolo, Falcidieno, Rossignac, ALGORITMICA 2004)

skeleton based on topological distance from curvature extrema

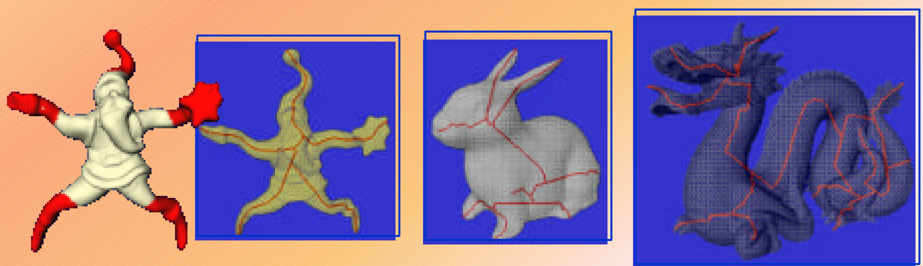
- Terminal nodes = RV
- Branching nodes = Split / Union vertices
- Arcs : given by topological adjacency





Skeleton based on topological distance from curvature extrema

- Curvature estimation is robust but time-consuming
- The skeleton is affine-invariant
- Meaningless for shape with no high-curvature features.



Skeleton based on topological distance from curvature extrema

☺ The starting point are chosen through a robust to noise multi-scale curvature evaluation

☺ It is useful for matching and reconstruction purposes

☹ the curvature evaluation is time consuming: $O(n^2)$

☺ The graph extraction from the starting point is cheap: $O(n)$

skeleton & Reeb graph

- let p be the centroid of a high-curvature region, we define $g_p(q)$:

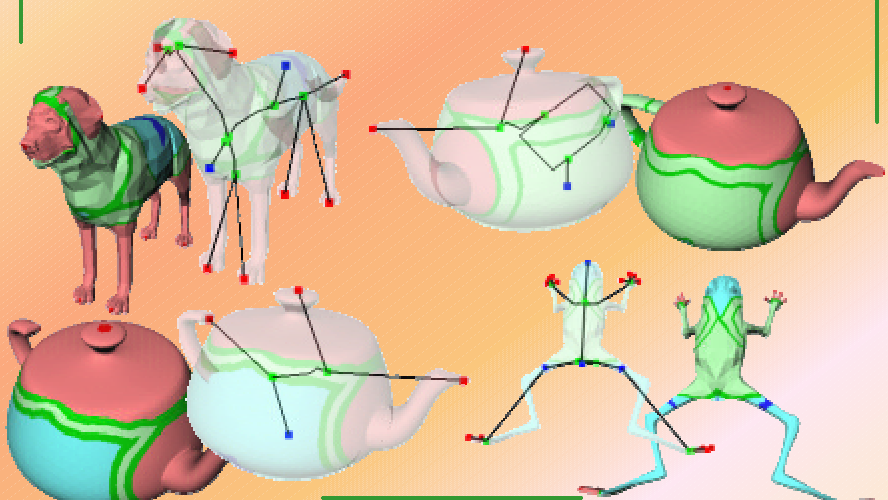
  $g_p(q)$:=*min{ k: q Î k-neighbourhood}*

- given $\{p_1,\ldots, p_n\}$ n centroids, we consider:

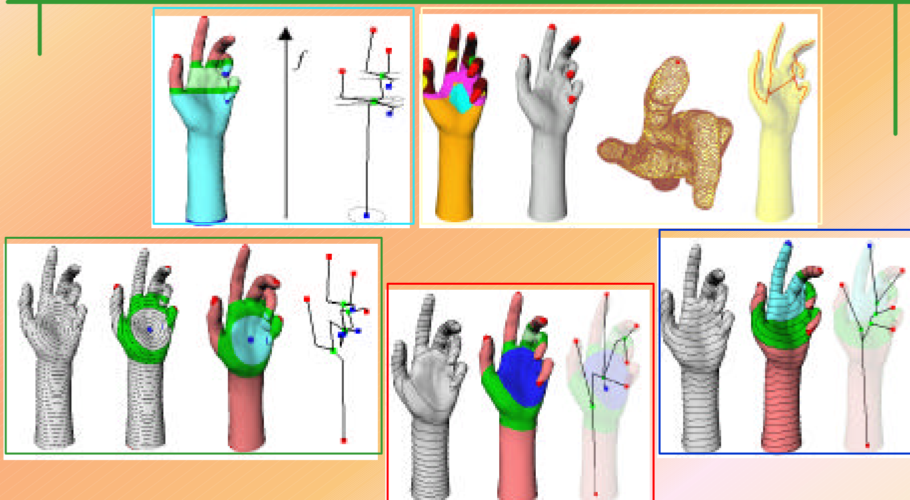
$$g(q) := \min\{g_{p_1}(q), \ldots, g_{p_n}(q)\}$$

- then: $f(q) := g_{\max} - g(q)$

examples



graph overview



overview

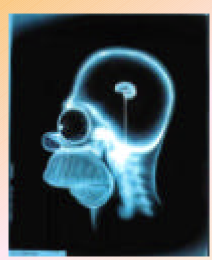
Functions	TASKS					COSTS	
	Recogn	Reconstr	Morph	Match	Anim	Worst	Average
Height function	😐	☺	☹	☹	☺	$O(1)$	$O(1)$
Barycentre dist.	☺	😐	☺	☹	☹	$O(n)$	$O(n)$
Geodesic dist.	☺	☹	☺	☺	😐	$O(n^2 logn)$	$O(bnlogn)$
Dist. from a source	😐	☺	☺	☹	☺	$O(nlogn)$	$O(nlogn)$
Top. Dist. from curvature seeds	☺	☺	☺	☺	☺	$O(n^2)$	$O(nk_{max})$

applications

- Morse theory
  - compression-simplification *(Bajaj&Schikore 1998,...)*
  - auditory display *(Axen& Edelsbrunner 1998,...)*
  - Morse-Smale decomposition *(Zomorodian et al. 2003, …)*
  - polygonalization of implicit surfaces *(Hart 1997, ...)*
- Reeb graph
  - reconstruction of surfaces from cross sections *(Shinagawa et al. 1991,...)*
  - simplification *(Biasotti et al. 2000, Wood et al 2004,...)*
  - shape similarity *(Hilaga et al. 2001, Tang&Schmitt 2004,…)* and shape matching *(Biasotti et al. 2003,…)*

Conclusions

- mathematics naturally provide a number of tools for describing either closed or bounded surfaces
- using real functions is useful to extract information on the overall shape of an object and to code it in a compact structure



Contacts:

silvia@ge.imati.cnr.it

patane@ge.imati.cnr.it