Global and Local Parameterization of Triangulated Surfaces

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Outline

- Global parameterization
  - problem statement & motivations
  - previous work
  - bordered 0-genus surfaces
  - bordered *g*-genus surfaces
- Surface analysis & shape-based local parameterization
  - patch decomposition
  - patch parameterization
- Conclusions

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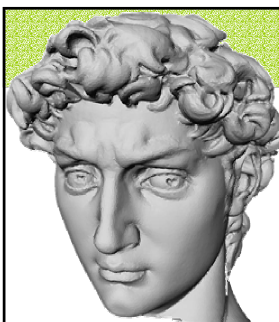
Triangle mesh



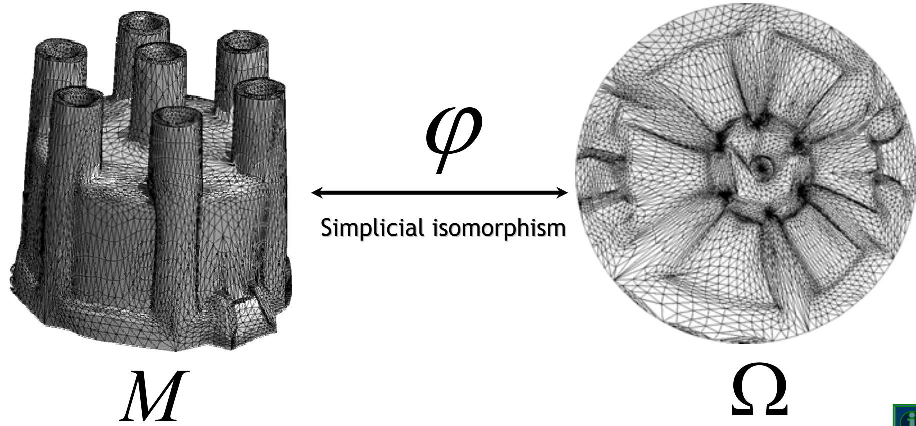
Surface=2-manifold

Representation: triangle mesh

? Find the (u,v) parameterization of *M*

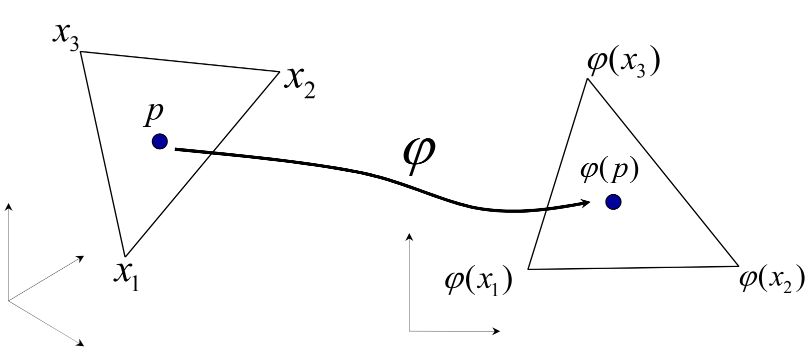
The global parameterization problem

Global Parameterization: continuous and piecewise linear isomorphism between the original surface and a planar region.



The global parameterization problem

φ is uniquely determined by its values on the vertices of *M*.

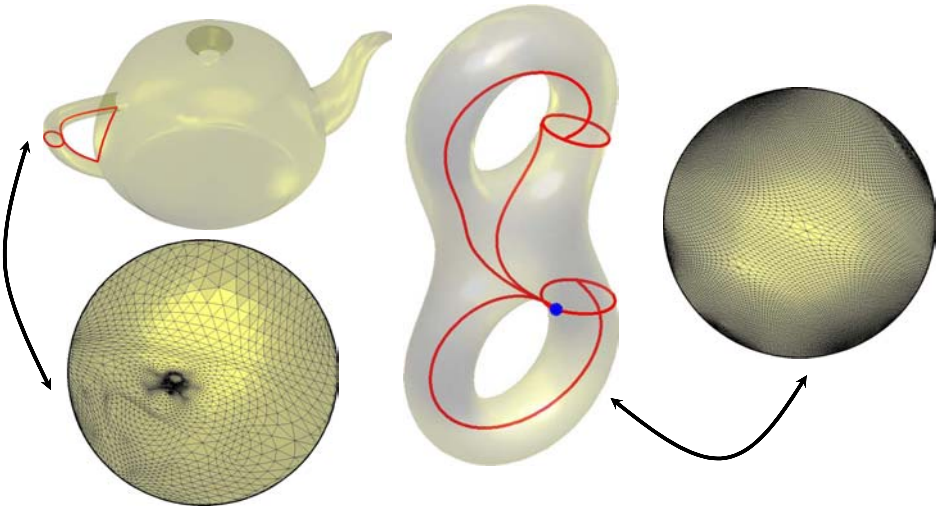


$$p = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \Leftrightarrow \varphi(p) = \lambda_1 \varphi(x_1) + \lambda_2 \varphi(x_2) + \lambda_3 \varphi(x_3)$$

$$\lambda_i \geq 0, i = 1,2,3, \lambda_1 + \lambda_2 + \lambda_3 = 1$$

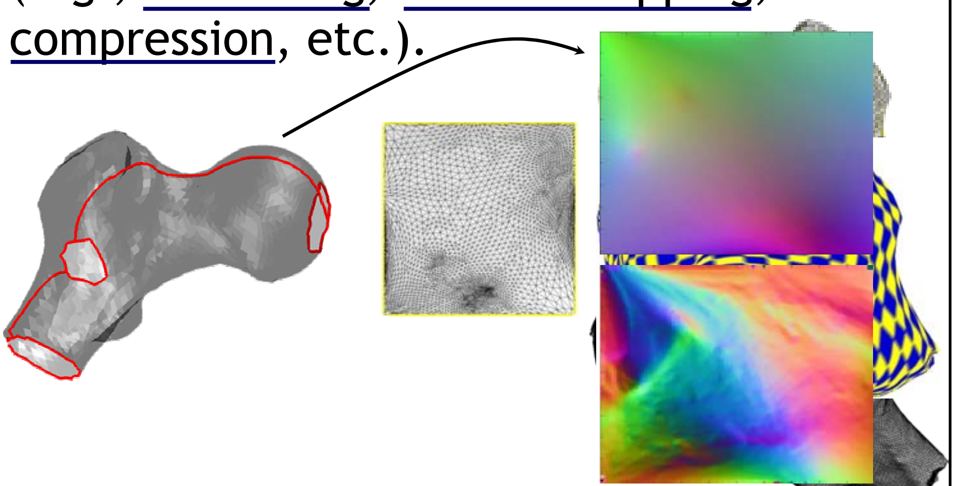
Why global parameterization?

Classification and computation of polygonal schema of arbitrary surfaces.



Why global parameterization?

Surface approximation and sampling (e.g., remeshing, texture mapping, compression, etc.).



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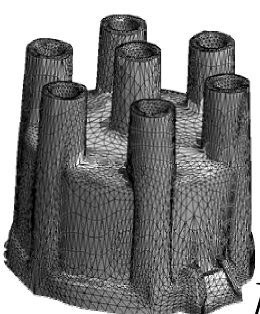
Parameterization of disc-like surfaces



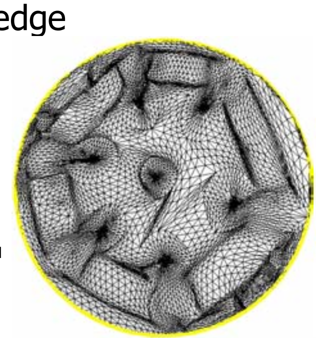
$$L = (l_{ij}) \in Gl_{n_V}(R), \ \sum_j l_{ij} = 1 \ \forall i = 1, \ldots, n_V$$

$$l_{ij} = \begin{cases} -1, i = j \\ > 0, (i, j) \text{ is an edge} \\ 0, \text{else} \end{cases}$$

M

$$Lx = b$$

Ω



shape preserving
least squares
constant weights

M. Floater. *Parameterization and smooth approximation of surface triangulations*. CAD 1997.

Related approaches

Most techniques vary on

- ✓ the distortion: they consider different definitions of energy functionals based on lengths, angles, and area (→ they mimic isometric, conformal, and area-preserving mappings);
- ✓ the numerical methods to find the "optimal":
  - ✓ conjugate gradient for linear systems;
  - ✓ gradient descent for non-linear energies.

DESBRUN01, LEE98; SHEFFER00, HORMANN99

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Parameterization of 0-genus surfaces

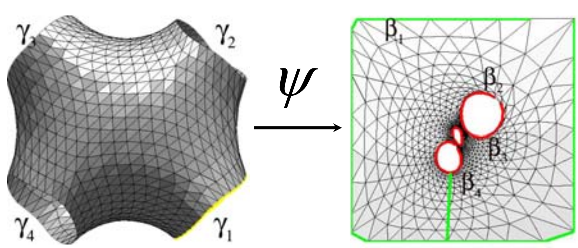
Proposition. Given a bordered 0-genus triangulated surface *M* with *k* boundary components, it is always possible to:

- ✓ map it onto a planar domain with *k* convex boundary components;
- ✓ to join them in linear time thus reducing *M* to a disc-like surface *M**;
- ✓ to have link paths among boundary components "independent" of the mesh connectivity and of class C^2.

G. Patanè, M. Spagnuolo, B. Falcidieno. *Para-Graph: Graph-based Parameterization of Triangle Meshes with Arbitrary Genus*. Computer Graphics Forum, 23-4, pp. 783-797, 2004.

Parameterization of 0-genus surfaces

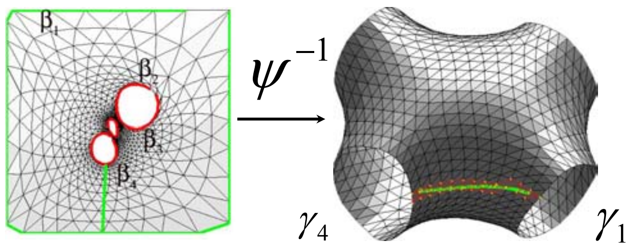
- ✓ Select one boundary component on *M*, e.g. the longer one γ_1;
- ✓ parameterize *M* w.r.t. γ_1;



- ✓ find the cut *l* of minimal length which joins β_1 to the closest internal boundary;

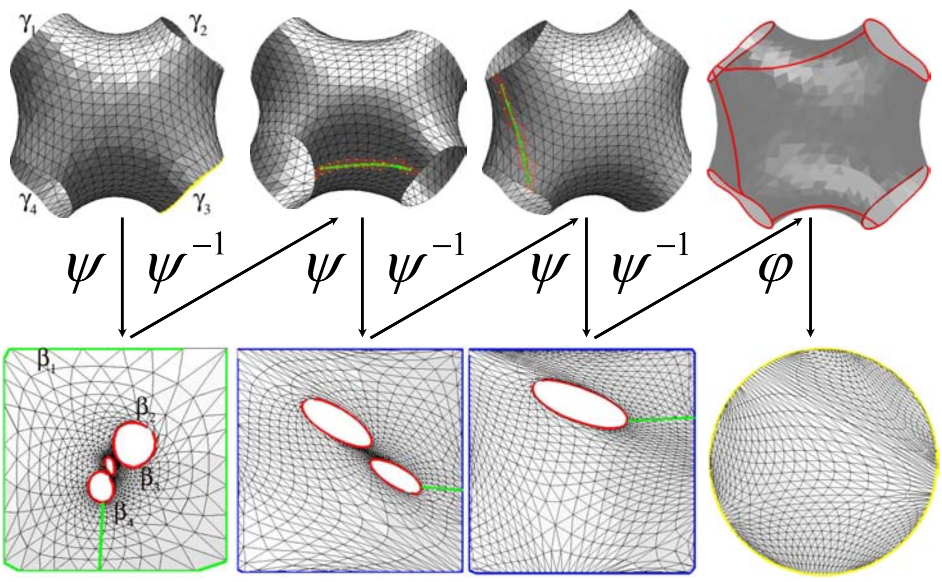
Parameterization of 0-genus surfaces

- ✓ map *l* back to *M* through ψ^{-1};



- ✓ iteratively apply the previous step (*k-1*)-times thus converting *M* to a disc-like surface *M**.

Parameterization of 0-genus surfaces



Cut characterization

Working on the parameter domain instead of *M*:

- ✓ no check of self-intersections among link paths;
- ✓ no use of approximated geodesic path → the cut is independent of the mesh connectivity;
- ✓ stable computation in $O(n)$ time;
- ✓ flexibility on the cut selection;
- ✓ useful for approaching the global parameterization problem of bordered surfaces with arbitrary genus.

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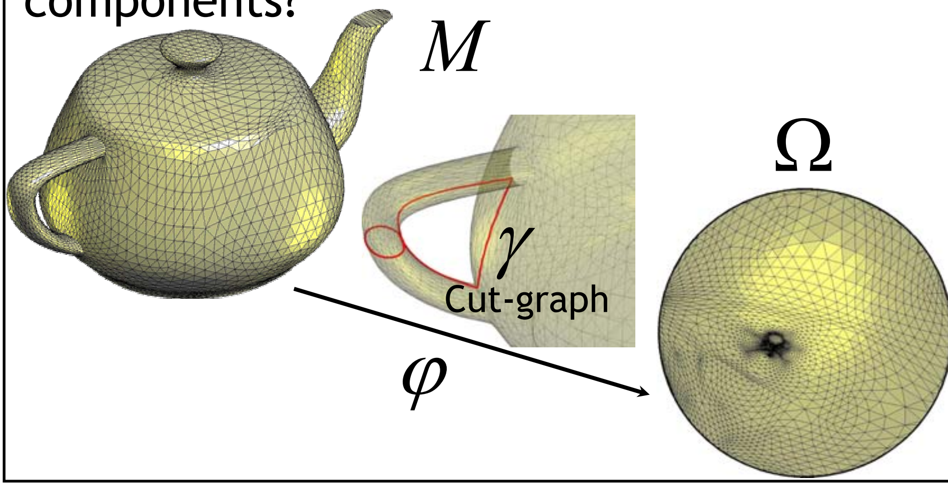
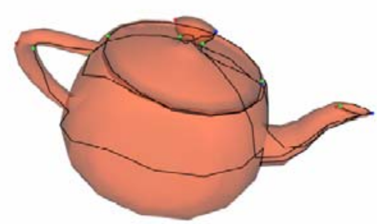
Parameterization of arbitrary surfaces

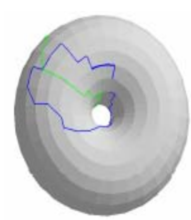
Generalization: how to deal with surfaces of arbitrary genus and boundary components?



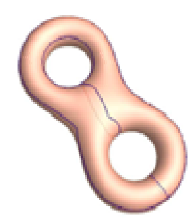
Parameterization of arbitrary surfaces



[Ni et al., SIGGRAPH 2004]



[Steiner et al., SM2004]



[Gu et al., SGP 2003]

General & open problems:

- ✓ cut smoothness;
- ✓ bordered surface analysis;
- ✓ simple approach;
- ✓ flexibility ←→ a family of cut graphs;
- ✓ …

Global parameterization of surfaces

- X. Ni, M. Garland, J. Hart. *Fair Morse Functions for Extracting the Topological Structure of a Surface Mesh*, SIGGRAPH 2004.
- D. Steiner, A. Fischer. *Planar Parameterization for Closed 2-Manifold Genus-1 Meshes*, ACM SM2004.
- X. Gu, S., Yau. *Global Conformal Surface Parameterization*, ACM SGP 2003.
- J. Erickson, S. Har-Peled. *Optimally Cutting a Surface into a Disk*, ACM SoCG 2002.
- D. Steiner, A. Fischer. *Cutting 3D Freeform Objects with Genus-n into Single Boundary surfaces Using Topological Graphs*, SM2002.
- ...

Parameterization of arbitrary surfaces

Let *M* be an arbitrary connected surface of genus *g≥1* and with *k* boundary components,

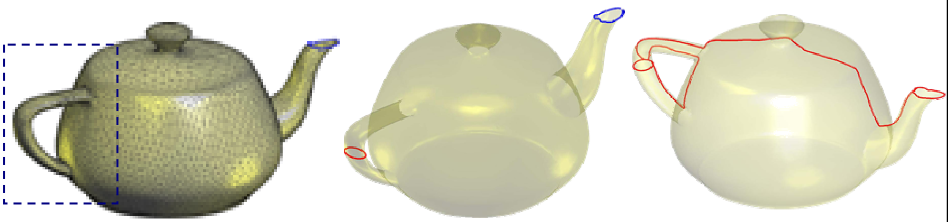
- *g* is evaluated in linear time through the Euler formula

$$g = \frac{1}{2}(2 - \chi(M) - k), \chi(M) = v - e + f.$$

- *g* is the maximal number of disjoint and non-separating loops of *M*.

Parameterization of arbitrary surfaces

- ✓ How to locate and cut the *g* topological handles of *M?*
- ✓ How to avoid to disconnect the surface?
- ✓ How to join the *(2g+k)* boundary components?



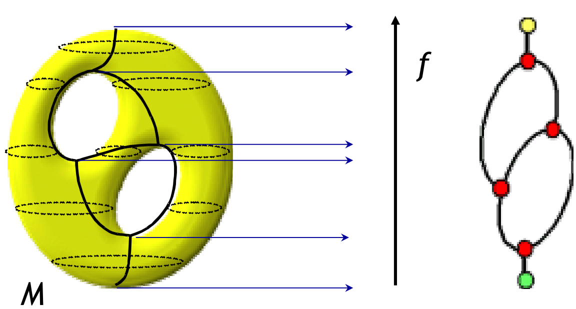
Handle location Handle cut Cut graph

Morse Theory and the Reeb Graph

Given *f: M*→*R*, the **Reeb graph** of *M* wrt *f* is the quotient space defined by "~" :

$$(x_1, f(x_1)) \sim (x_2, f(x_2)) \Leftrightarrow f(x_1) = f(x_2)$$

and x_1 and x_2 belong to the same connected component of $f^{-1}(f(x_1))$.

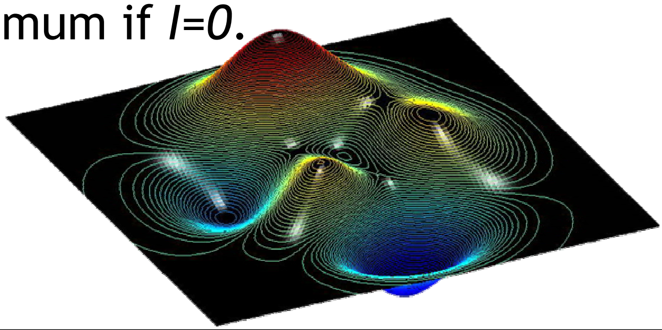


✓J. Milnor. *Morse Theory*. Princeton University Press, 1963.

Morse Theory and the Reeb Graph

If *p* is a critical (i.e., $\nabla f(p)=0$) and non-degenerate (i.e., $det(H_f(p))\neq 0$) point of (M,f) and $I:=index(H_f(p))$, then

- ✓ *p* is a maximum if $I=2$;
- ✓ *p* is a saddle if $I=1$;
- ✓ *p* is a minimum if $I=0$.

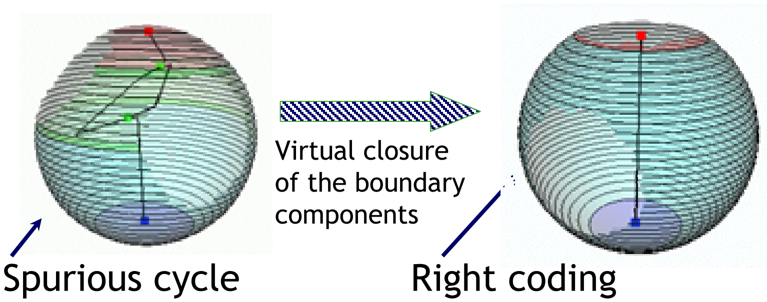


Morse Theory and the Reeb Graph

Proposition. Let *M* be a connected orientable 2-manifold of genus *g*, $f: M \rightarrow R$ a Morse function, and *G* the Reeb graph of (M,f). Then,

- ✓ $\chi(M)$=maxima-saddles+minima;
- ✓ if *M* is closed, *G* has *g* loops;
- ✓ if *M* has *k* boundary components, *G* has *l* loops with $g \leq l \leq 2g+k-1$.

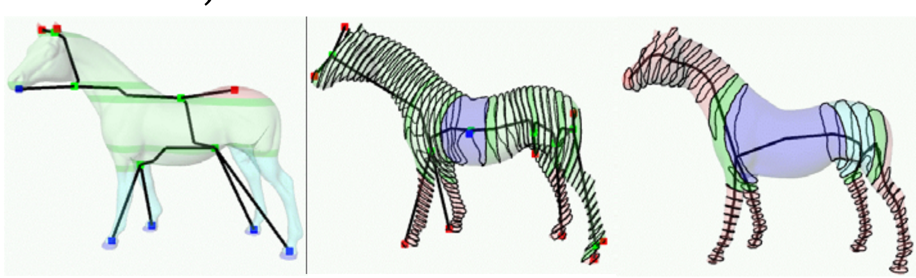
Morse Theory and the Reeb Graph



S. Biasotti. *Reeb graph of bordered triangle meshes*. Shape Modeling International, 2004.

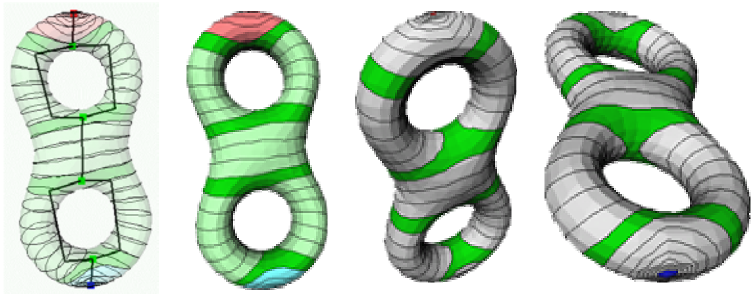
Morse Theory and the Reeb Graph

Various *f*: height function, geodesic distance from curvature extrema, harmonic, …

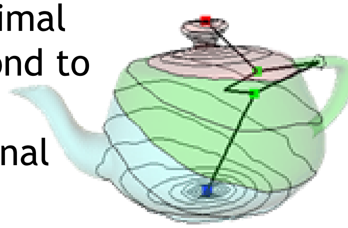


- S. Biasotti, S. Marini, M. Mortara, G. Patanè. *An overview on properties and efficacy of skeletons in Shape Modeling*. In Proceed. SMI 2003, pp. 245-254, IEEE Computer Society Press.
- M. Mortara, G. Patanè. *Shape covering for skeleton extraction*. International Journal of Shape Modelling, 8-2, pp. 139-158, 2002.

Morse Theory and the Reeb Graph



Harmonic scalar fields with a minimal number of critical points correspond to



- ✓ smooth iso-contours;
- ✓ a Reeb graph with only 2 terminal nodes.

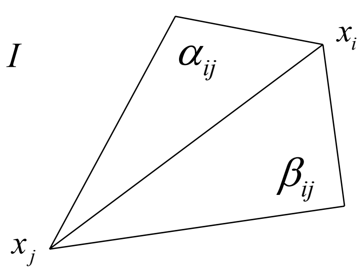
✓ X. Ni, M. Garland, J.C. Hart. *Fair Morse function for extracting the topological structure of a surface mesh*. SIGGRAPH 2004.

Morse Theory and the Reeb Graph

A harmonic scalar field $f : M \rightarrow R$ is the solution of the following linear system:

$$\begin{cases} \sum_{j \in N(i)} w_{ij}(f(x_j) - f(x_i)) = f(x_i), i \in I \\ f(x_i) = \alpha_i, i \in B \end{cases}$$

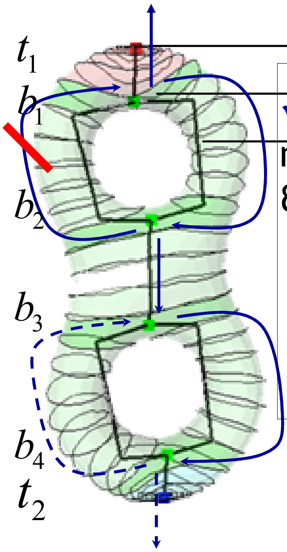
$$w_{ij} := \cot(\alpha_{ij}) + \cot(\beta_{ij})$$



- ✓ if all the constrained minima and maxima are assigned the same global values *(m,M)*, then the constraints are the unique extrema of *f* (→ "<u>optimality</u>": 1 min, 1 max, 2g saddles).

Parameterization of arbitrary surfaces

Handle identification & cut



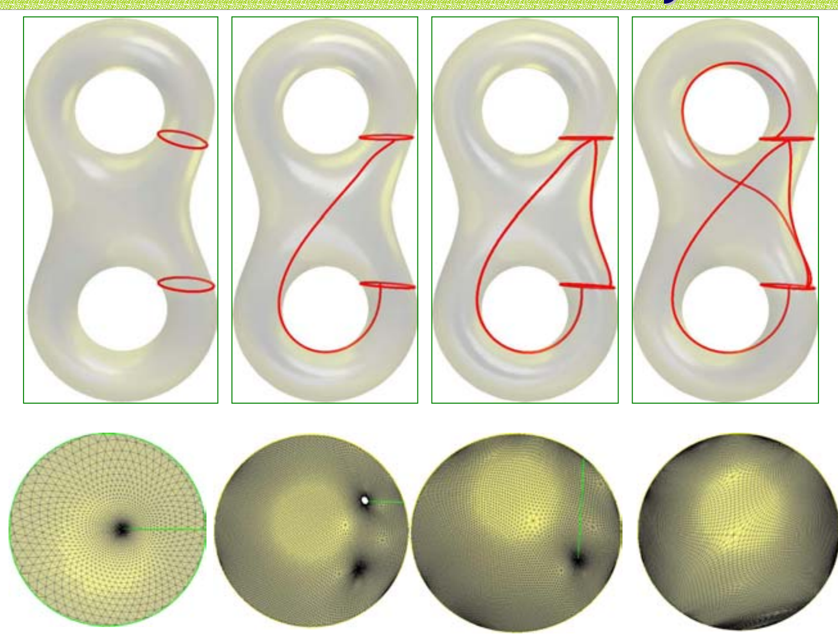
✓ Internal path between branching nodes: cut iff *M* is not disconnected & g→(g-1);

YES (cut=iso-contours of *M* w.r.t. *f*)

✓ mark the internal nodes of the current arc as visited.

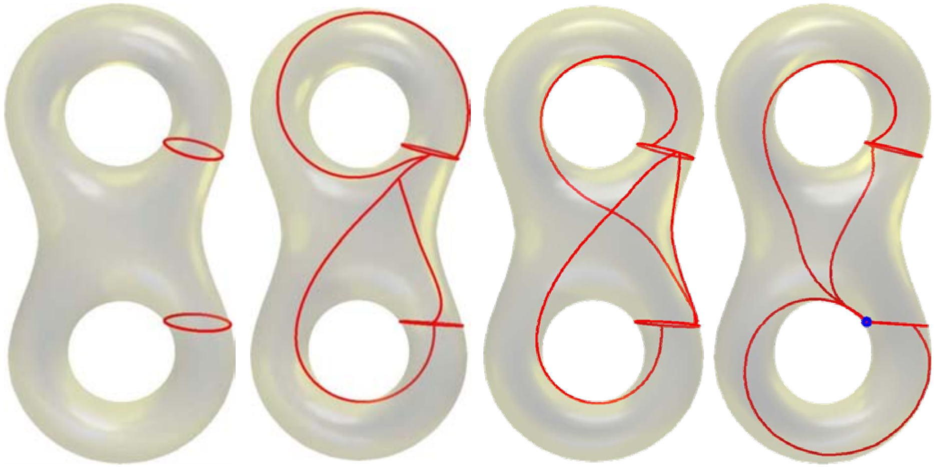
Repeat the iteration until g cuts have been performed.

Parameterization of arbitrary surfaces

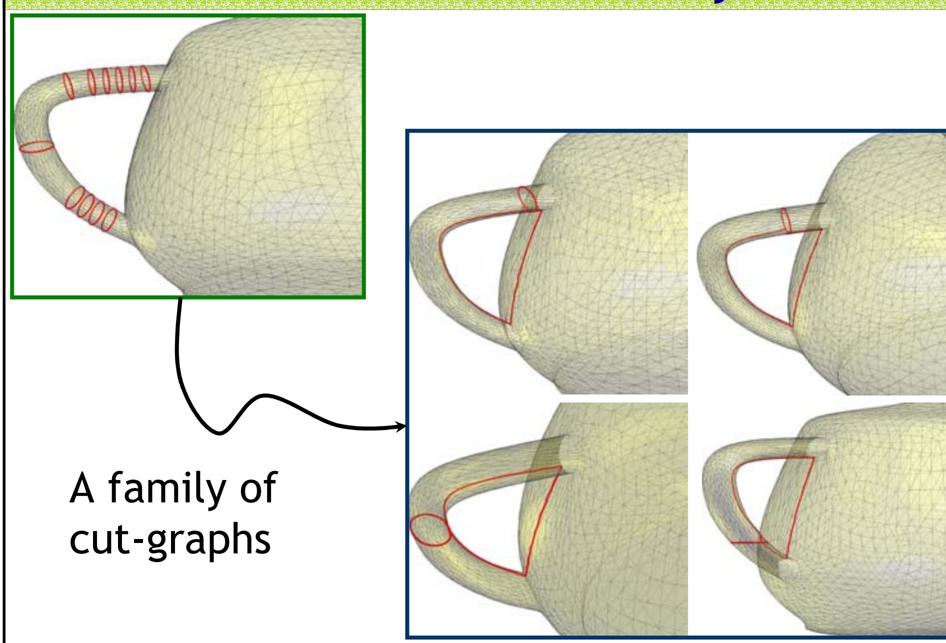


Parameterization of arbitrary surfaces

Output: a family of possible meridian cuts for the topological handles of M.

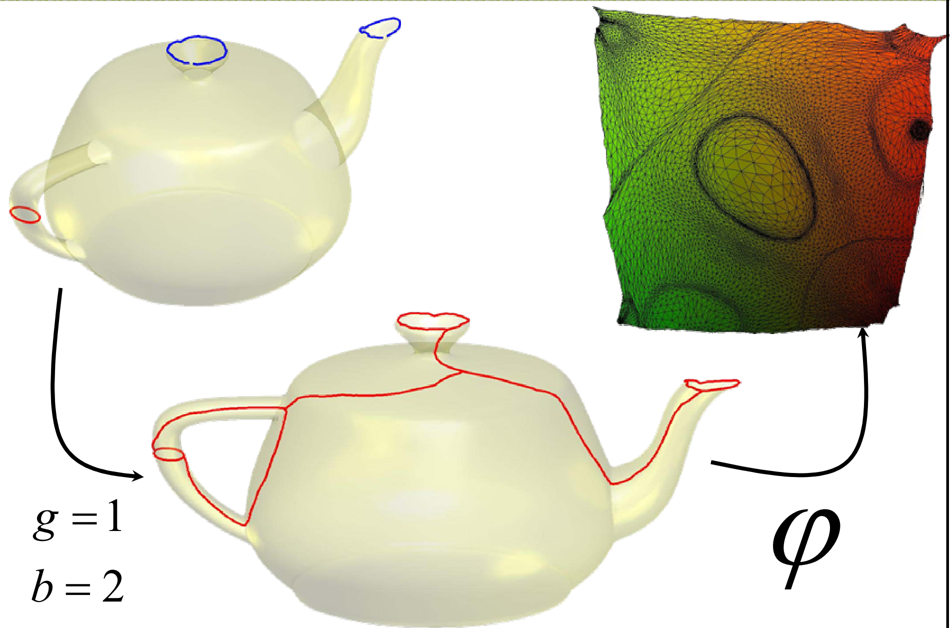


Parameterization of arbitrary surfaces



A family of cut-graphs

Parameterization of arbitrary surfaces



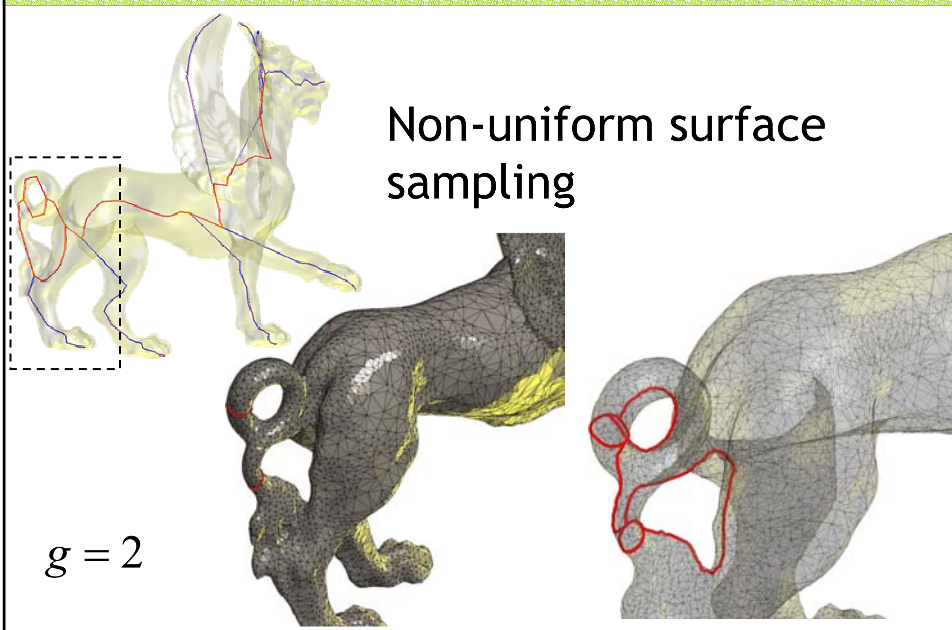
$g = 1$

$b = 2$

φ

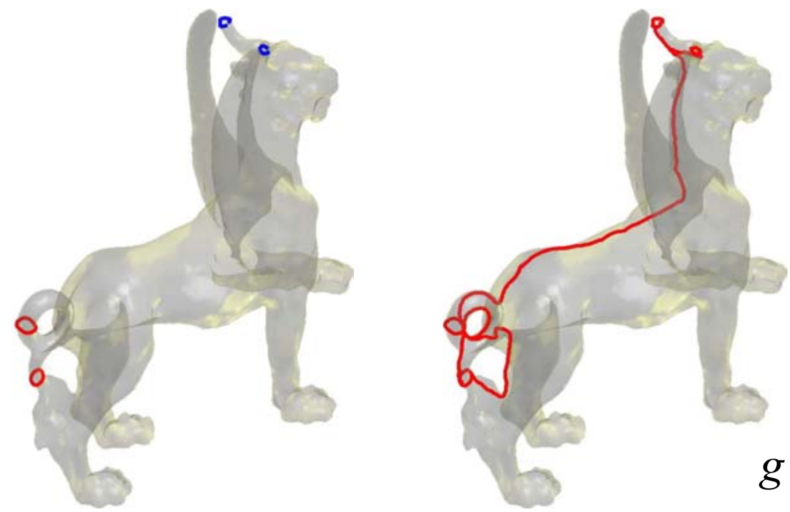
Parameterization of arbitrary surfaces

Non-uniform surface sampling



$g = 2$

Parameterization of arbitrary surfaces

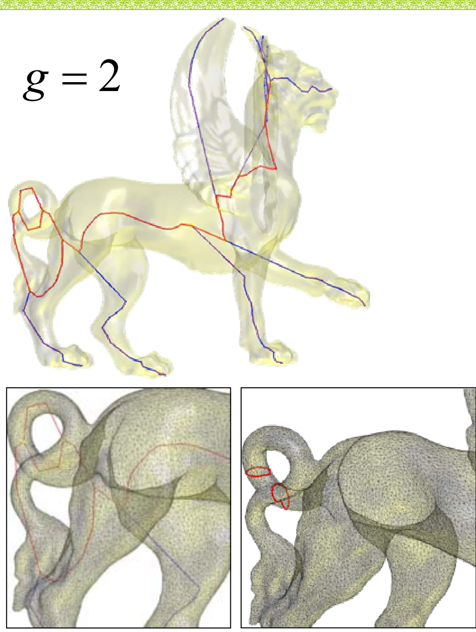


$g = 2$

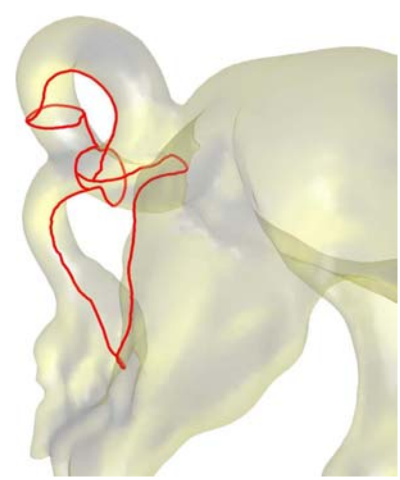
$b = 2$

Parameterization of arbitrary surfaces

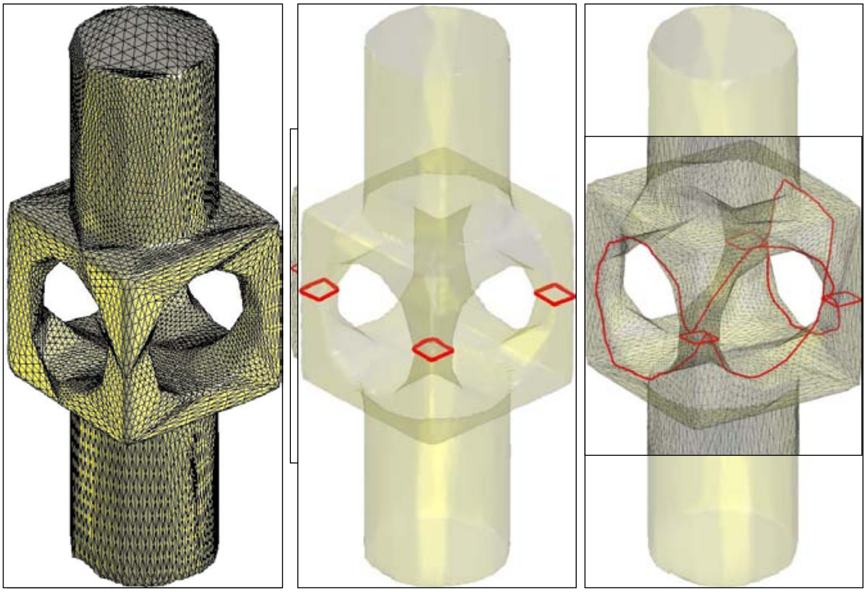
$g = 2$



Uniform sampling



Parameterization of arbitrary surfaces



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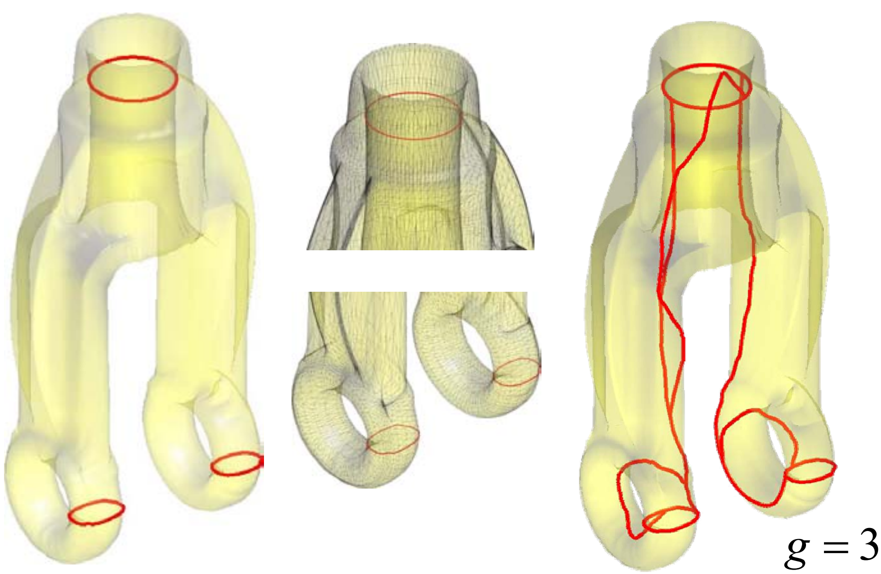

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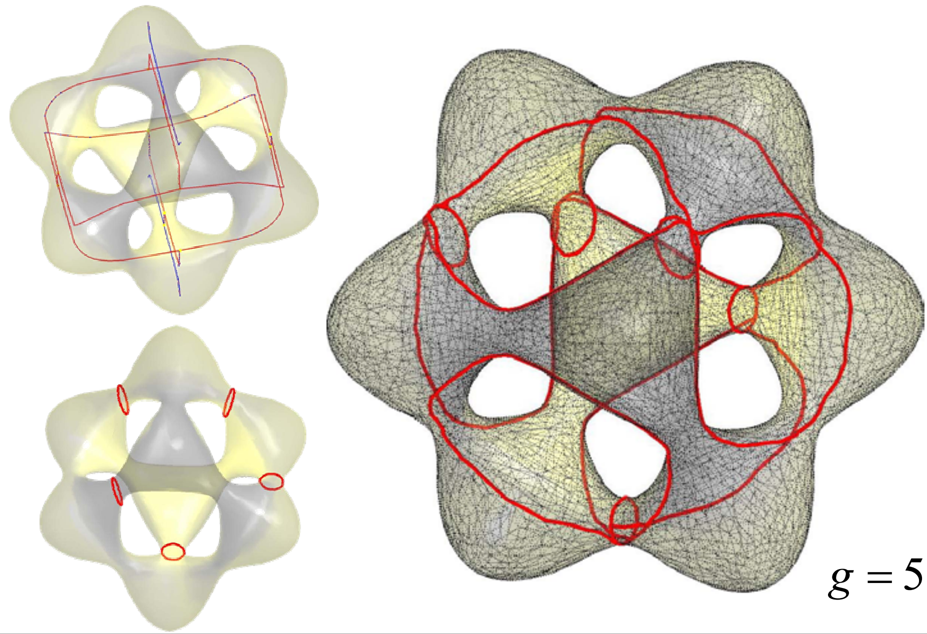

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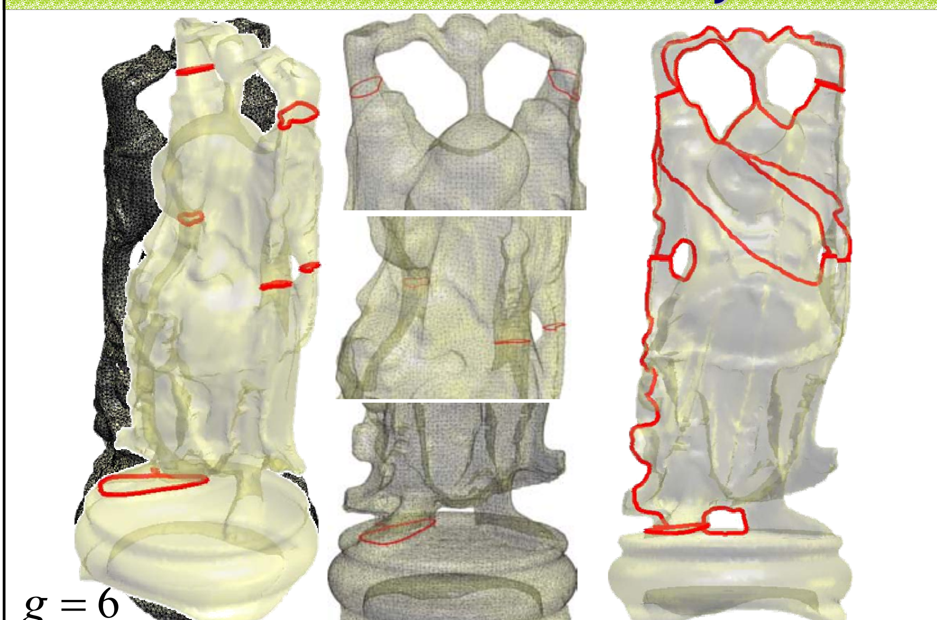
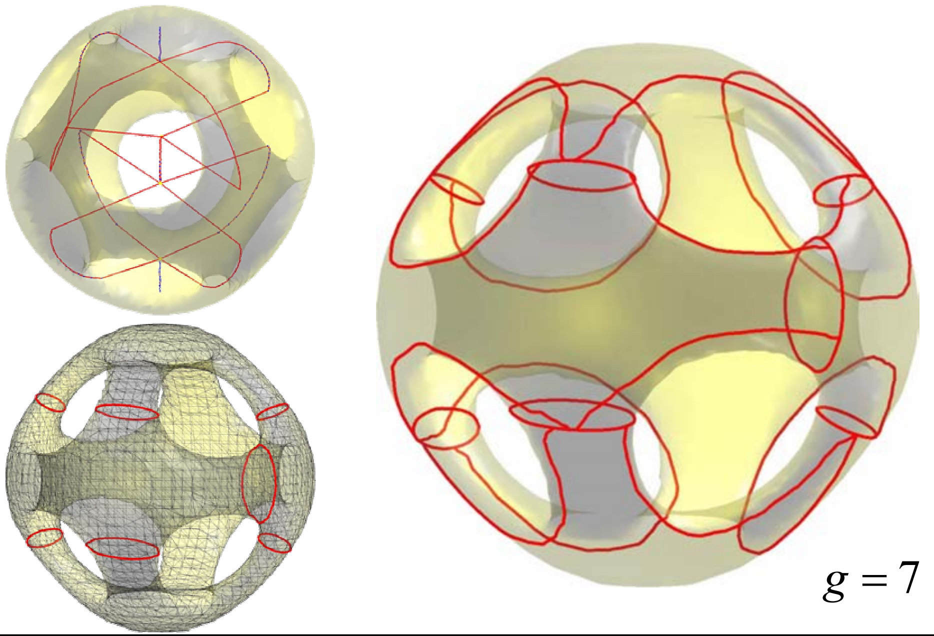


Parameterization of arbitrary surfaces



Global Parameterization

Proposition. Let *M* be an arbitrary surface *M*. If

- ✓ $g=0$, the cut graph is reduced and evaluated in $O(n)$ time;
- ✓ $g\geq 1$, the cut graph has $2g$ redundant edges and it is evaluated in $O(n\log n)$ time.

G. Patanè. *Global parameterization of bordered triangle meshes with arbitrary genus*. IMATI-GE/CNR Technical Report 2005.

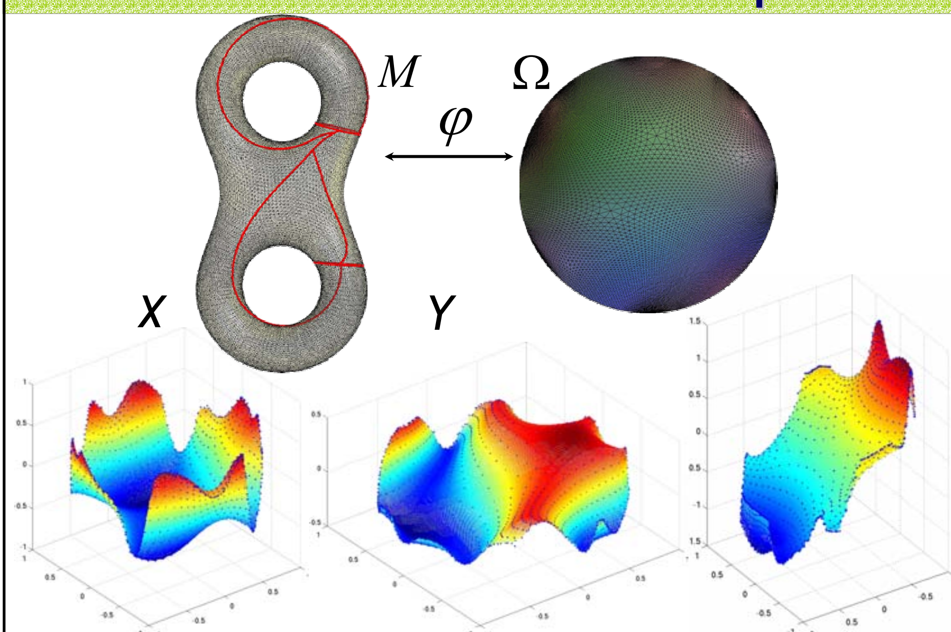
Parameterization of arbitrary surfaces

Proposition. Let M be an arbitrary surface, γ and β two cuts which reduce M to a disk-like surface, φ_γ and φ_β the related unfoldings onto the plane. Then,

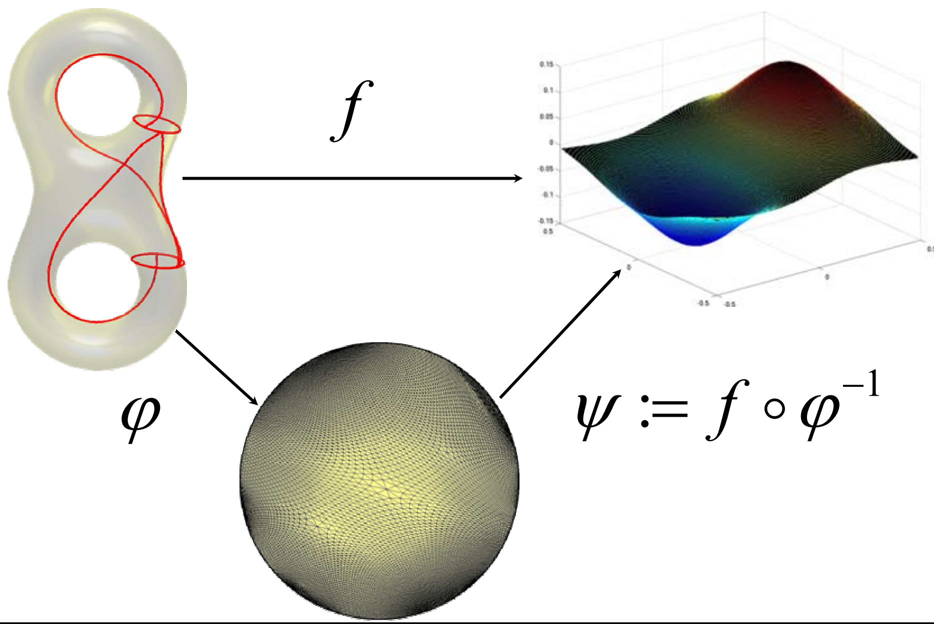
$$\frac{\|(\varphi_\gamma(x_i))_i - (\varphi_\beta(x_i))_i\|_2}{\|(\varphi_\gamma(x_i))_i\|_2} \le \kappa_2(W)\,\|b_\gamma - b_\beta\|_2$$

with W sub-matrix of L not affected by the cuts and $\|b_\gamma - b_\beta\|_2$ discrepancy on the boundary conditions.

Surface interpolation



Analysis and modelling of scalar fields

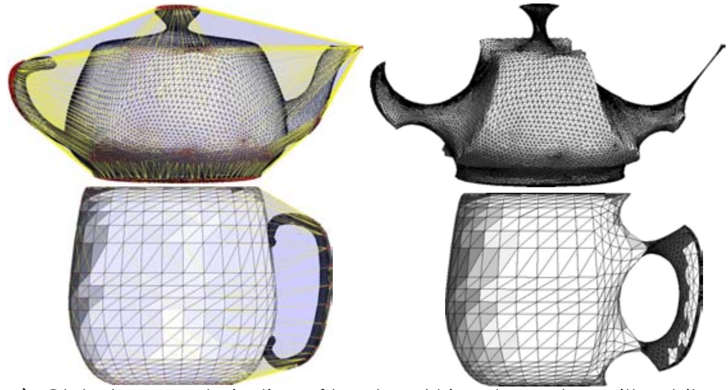


Parameterization of arbitrary surfaces

- ✓ Flexibility: family of cut graphs;
- ✓ generality: bordered and closed surfaces are treated with a unique approach;
- ✓ smoothness and independence of the mesh connectivity;
- ✓ computational cost: *O(n)* if *g=0*, *O(nlogn)* otherwise;
- ✓ constant redundancy of the cut graph.

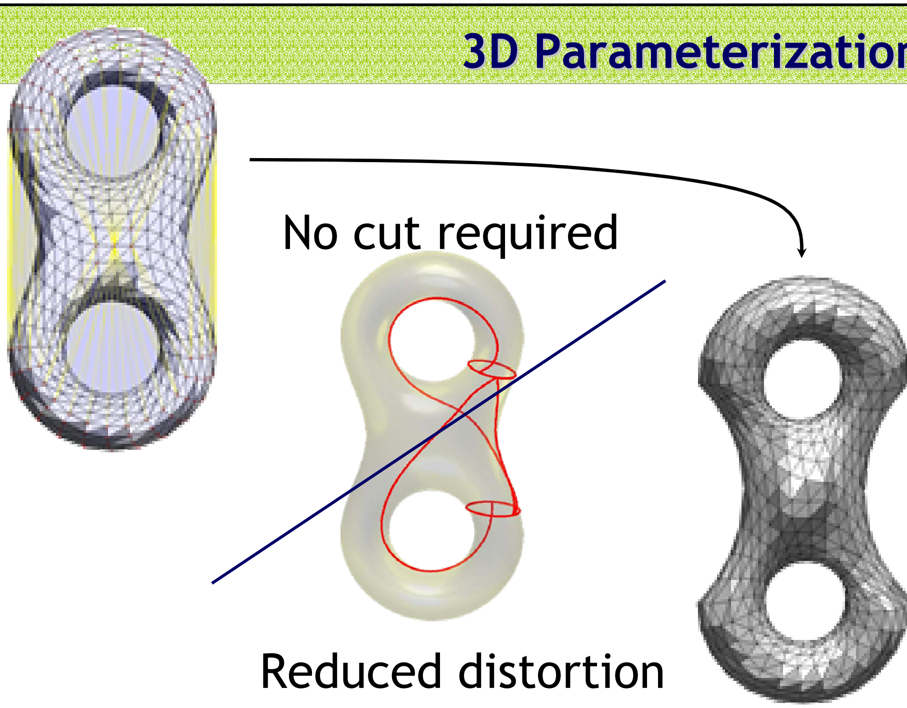
3D Parameterization

Proposition. Any arbitrary surface *M* can be embedded onto a *3D* domain *P* which belongs to the convex hull of *M*.



G. Patanè. *Global parameterization of bordered triangle meshes with arbitrary genus*. IMATI-GE/CNR Technical Report.

3D Parameterization



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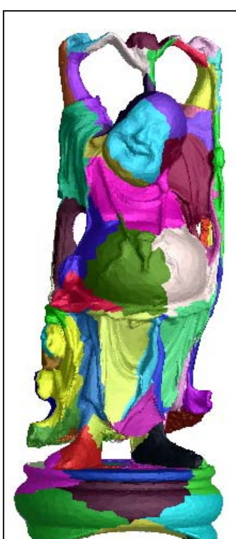
- ✓ Conclusions

Local parameterization



[Images from Gu2002]

PIPELINE

- ✓ Decompose the input surface into disk-like patches;
- ✓ parameterize each patch.

PRO

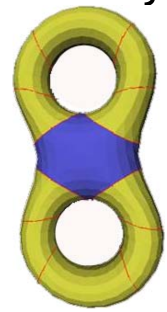
- ✓ Simple implementation;
- ✓ no assumptions on the surface genus or point density;
- ✓ well-studied.

CONS

- ✓ Patch symmetry can be missed;
- ✓ patches do not reflect the curvature distribution;
- ✓ remesh requires to control smoothness along boundaries;
- ✓ extraordinary vertices of the remeshed dataset are scattered on the whole surface.

Local parameterization

Partition of the input surface into a family of *m* patches $R_i, \ldots, R_m$ such that



$\bigcup_{i=1}^{m} R_i = M;$

R_i is a connected region;

$R_i \cap R_j = \emptyset, i \neq j;$

$R_i \neq \emptyset$ has 0-genus and an arbitrary number of boundary components.

M. Mortara, G. Patanè, G., M. Spagnuolo, B. Falcidieno, J. Rossignac. *Blowing Bubbles for the Multi-scale Analysis and Decomposition of Triangle-Meshes*. ALGORITHMICA 2004.

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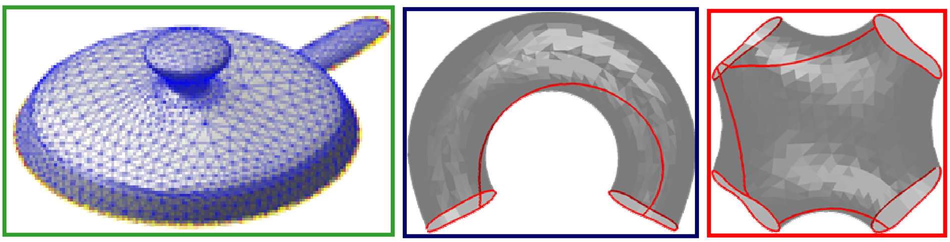
  - ✓ bordered *g*-genus surfaces
  - ✓ 3D parameterization
- ✓ Shape-based local parameterization
  - ✓ patch decomposition
  - ✓ patch parameterization
- ✓ Conclusions

Local parameterization

Only three types of patches: R_i is

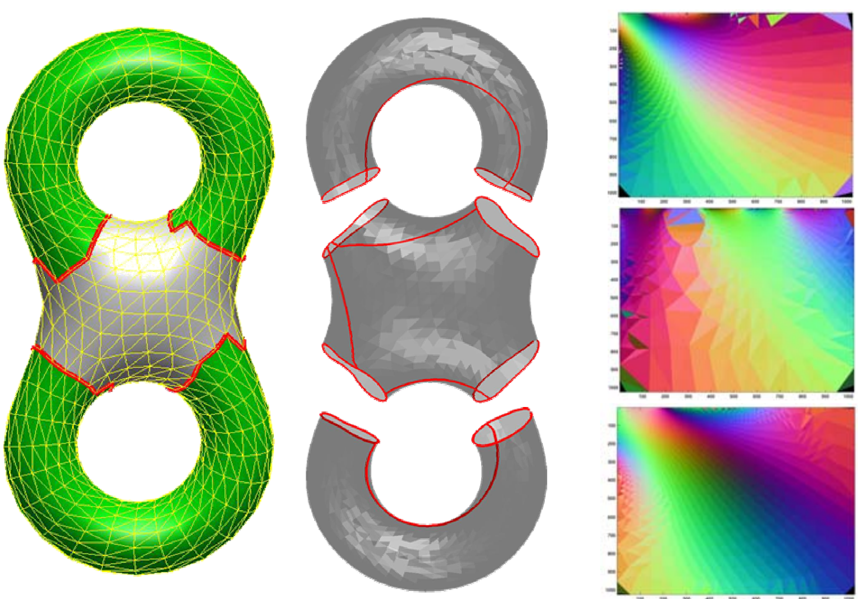
- ✓ **conical** if it has 1 boundary;
- ✓ **cylindrical** if it has 2 boundary components;
- ✓ **body type** otherwise.

<u>to be cut</u>

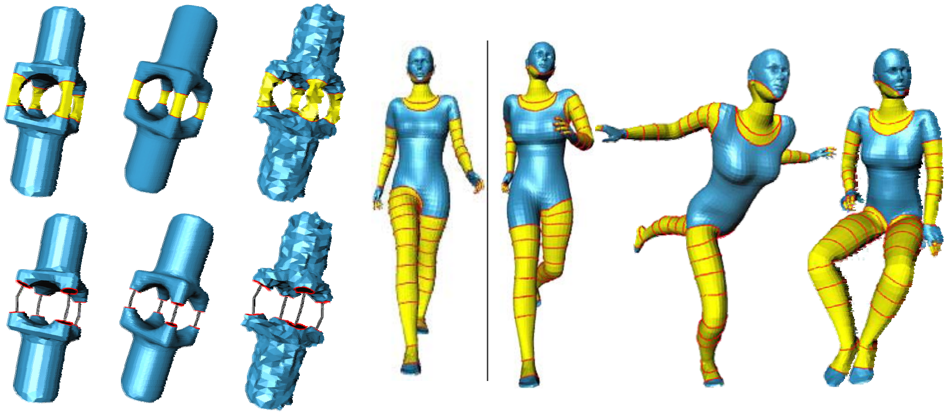


G. Patanè, M. Spagnuolo, B. Falcidieno. *Para-Graph: Graph-based Parameterization of Triangle Meshes with Arbitrary Genus*. Computer Graphics Forum, 23-4, pp. 783-797, 2004.

Local parameterization



Parameterization of arbitrary surfaces



M. Mortara, G. Patanè, G., M. Spagnuolo, B. Falcidieno, J. Rossignac. *Plumber: a method for the multi-scale decomposition of 3D shapes into tubular primitives and bodies*. ACM SOLID MODELLING 2004.

Local parameterization

REEB GRAPH INDUCED BY DIFFERENT MAPPING FUNCTIONS

- ✓ Y. Shinagawa and T. Kunii. *Constructing a Reeb graph automatically from cross sections*, IEEE Computer Graphics and Applications, 11(5):44-51, 1991.
- ✓ M. Hilaga, Y. Shinagawa, T. Komura and T. Kunii. *Topology matching for fully automatic similarity estimation of 3D shapes*. Proc. SIGGRAPH 2001, pp. 203-212.
- ✓ T. K. Dey, J. Giesen, S. Goswami. *Shape segmentation and matching with flow discretization*. Int. Workshop on Algorithms and Data Stracture, 2003.

MULTI-RESOLUTIVE CURVATURE-BASED SEGMENTATION

- ✓ M. Mortara, G. Patanè, G., M. Spagnuolo, B. Falcidieno, J. Rossignac. *Plumber: a method for the multi-scale decomposition of 3D shapes into tubular primitives and bodies*. ACM SOLID MODELLING 2004.
- ✓ M. Mortara, G. Patanè, M. Spagnuolo, B. Falcidieno, J. Rossignac. *Blowing bubbles for multi-scale analysis and decomposition of triangle meshes*. In Algorithmica, Special Issue on Shape Algorithmics, Volume 38, Num. 2, 2003, pp. 227-248, Springer-Verlag.

VERTEX CLUSTERING

- ✓ S. Katz, A. Tal. *Hierarchical mesh decomposition using fuzzy clustering*. SIGGRAPH 2003.
- ✓ ...

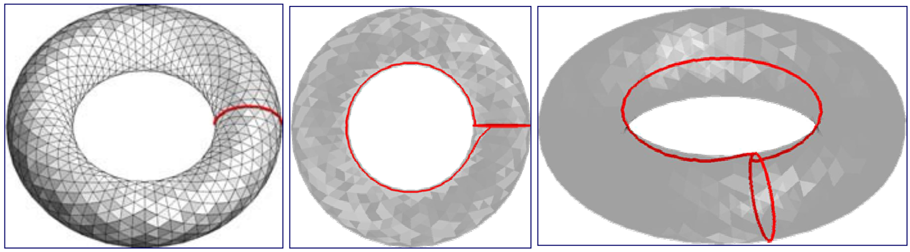
Outline

- ✓ Global parameterization
  - ✓ problem statement & motivations
  - ✓ previous work
  - ✓ bordered 0-genus surfaces
  - ✓ bordered *g*-genus surfaces
  - ✓ 3D parameterization
- ✓ Shape-based local parameterization
  - ✓ patch decomposition
  - ✓ patch parameterization
  - ✓ shape-graph
- ✓ Conclusions

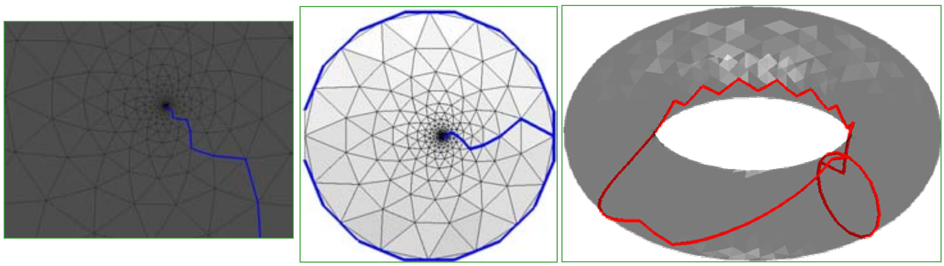
Future work

- ✓ Find meridians and longitudes of the topological handles (i.e., homology bases) ←→ constrained parameterization of bordered 0-genus surfaces;
- ✓ editing through cut & past of surface;
- ✓ 3D surface parameterization;
- ✓ surface approximation & compression;
- ✓ ...

Parameterization of arbitrary surfaces



Approximated vs exact geodesic cuts



Simplicial complex & maps

A finite family *K* of simplices is a **simplicial complex** iff:

- contains all the faces of each simplex;
- the intersection of any pair of simplices is a common face.

A map $\varphi : K \rightarrow L$ is a **simplicial map** iff

$$\varphi(\Delta(a_0,...a_n)) = \Delta\{\varphi(a_0),...,\varphi(a_n)\}$$