# Topologia computazionale per la descrizione di forma

*Matematica, Forme, Immagini*

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- exploring shapes measuring the geometric properties of a real function
  - Morse theory
  - critical points
  - computational topology
- shape encoding based on differential properties of real functions
  - Reeb graphs
- application to shape matching and future perspectives

# mathematics provides a formal framework to

- ✓ abstract

  to eliminate and discard redundant information and keep what is more **significant**

- ✓ decompose

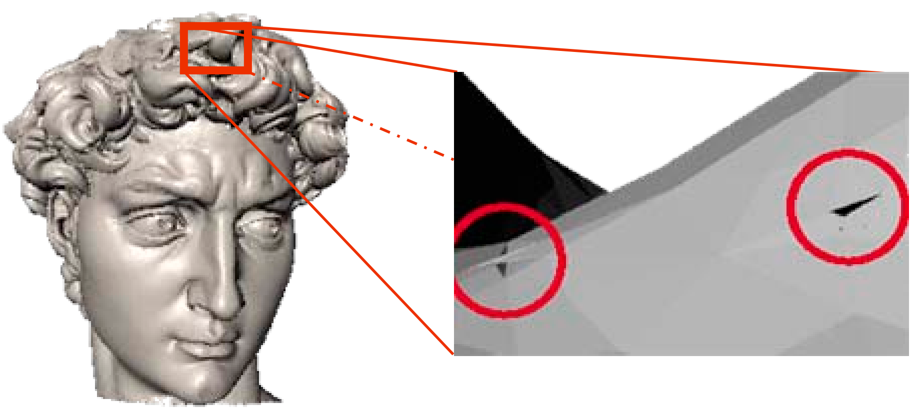
  to segment a model into subparts each simpler that the original

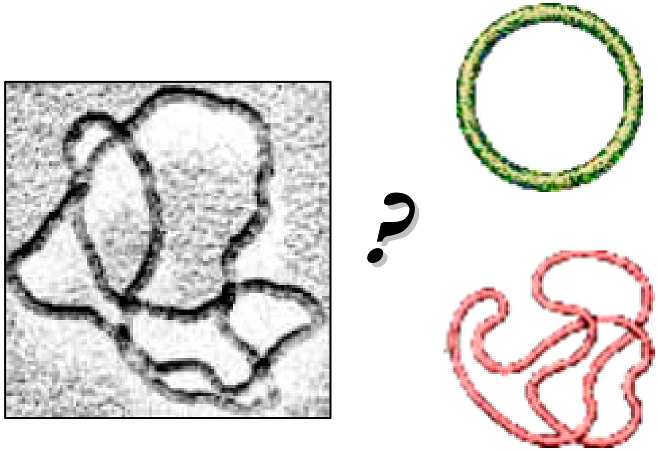
  **structural decompositions** code salient features along with their relations

- ✓ classify

  to **identify** a model as a member of some class of objects

- topology attempts to understand the global connectivity of an object by considering its local connectivity
- it concerns how an object is placed in another space
- it gives a flexible framework suitable for many applications





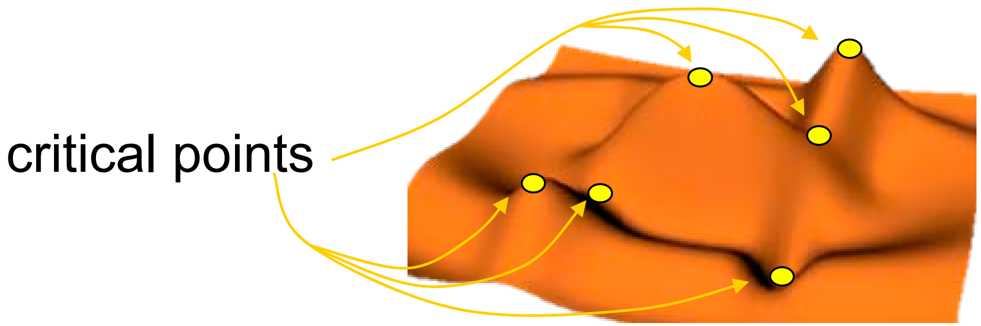
- aim
  - to code the main shape features
  - to define a conceptual model mathematically well-defined and related to the significant shape features
  - to provide a topology-driven data simplification
  - to structure the shape into a topological skeleton
  - to be computationally affordable

- "computational topology focuses on computational aspects of problems with a topological flavour"

- "identification and formalisation of topological questions in computer applications and the study of algorithms for solving them"

(Dey, Edelsbrunner, Guha 1999) (Hart 1999)

- ✓ Morse theory studies the relationship between a function's critical points and the topology of its domain (Milnor, 1963)
  - the perception of shape is focused at maxima, saddles and minima




- ✓ it indicates when the topological type changes and what kind of changes take place
- ✓ it provides a surface decomposition into a limited set of primitive topological cells, defined by the surface critical points and their corresponding index
  - the index of a critical point is the number of negative eigenvalues of its Hessian

- in other words:
  - a point *p* is ***critical*** for *f* if:

$$\frac{\partial f}{\partial x_1}(p)=0, \frac{\partial f}{\partial x_2}(p)=0, \ldots, \frac{\partial f}{\partial x_k}(p)=0$$

  - *f* is ***Morse*** at *p* if:

$$\left|H_f(p)\right| = \left|\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right| \neq 0.$$

- the definition of critical points is *local* and *sensitive* to small perturbations of the surface.
- the function *f* is called ***simple*** if any pair *x*, *y* of distinct critical points verifies $f(x) \neq f(y)$.

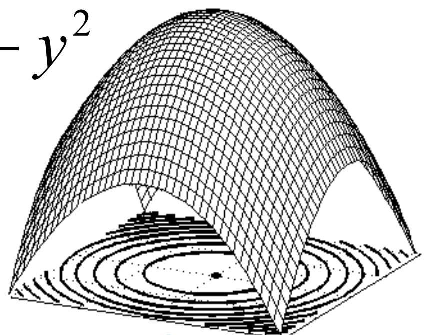
# Morse theory & critical points

- **Morse Lemma**: In a neighbourhood of a critical point *p*, a Morse function *f* can be expressed as:

$$f = f(p) - (y_1)^2 - \ldots - (y_\lambda)^2 + (y_{\lambda+1})^2 + \ldots (y_n)^2$$

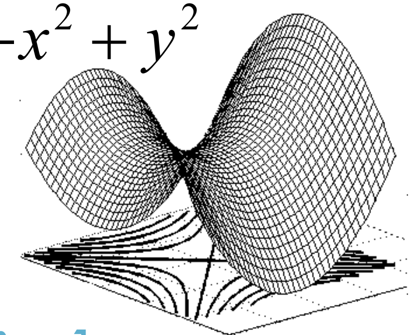
where $\lambda$ is the *index* of the critical point.

$f = -x^2 - y^2$



maximum $\lambda$=2

$f = -x^2 + y^2$



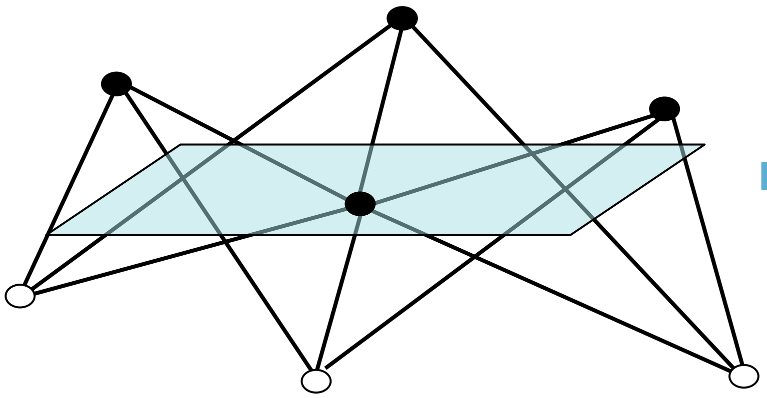
saddle $\lambda$=1

- **Euler formula**: *#maxima - #saddles + #minima=* $\chi(S)$

- if *f* is general, they may be detected by analysing the vertex star *(Banchoff 1967,1970, … )*
  - a discrete index is defined as

$$i(v,f) = 1 - \frac{1}{2}\#\{(v_1,v_2) \subset link(v) \mid (f(v_1)-f(v))*(f(v)-f(v_2)) < 0\}$$

  - critical points have index different from 0



multiple saddle

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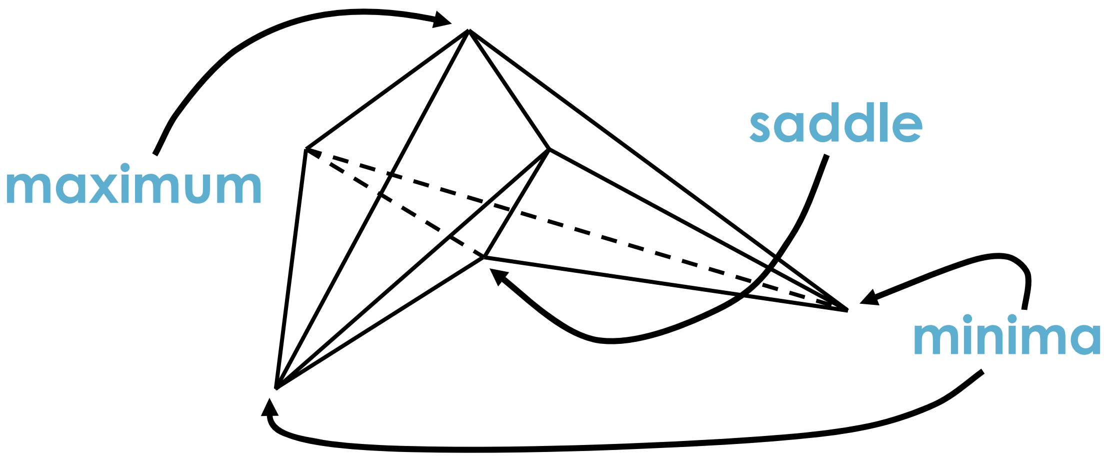


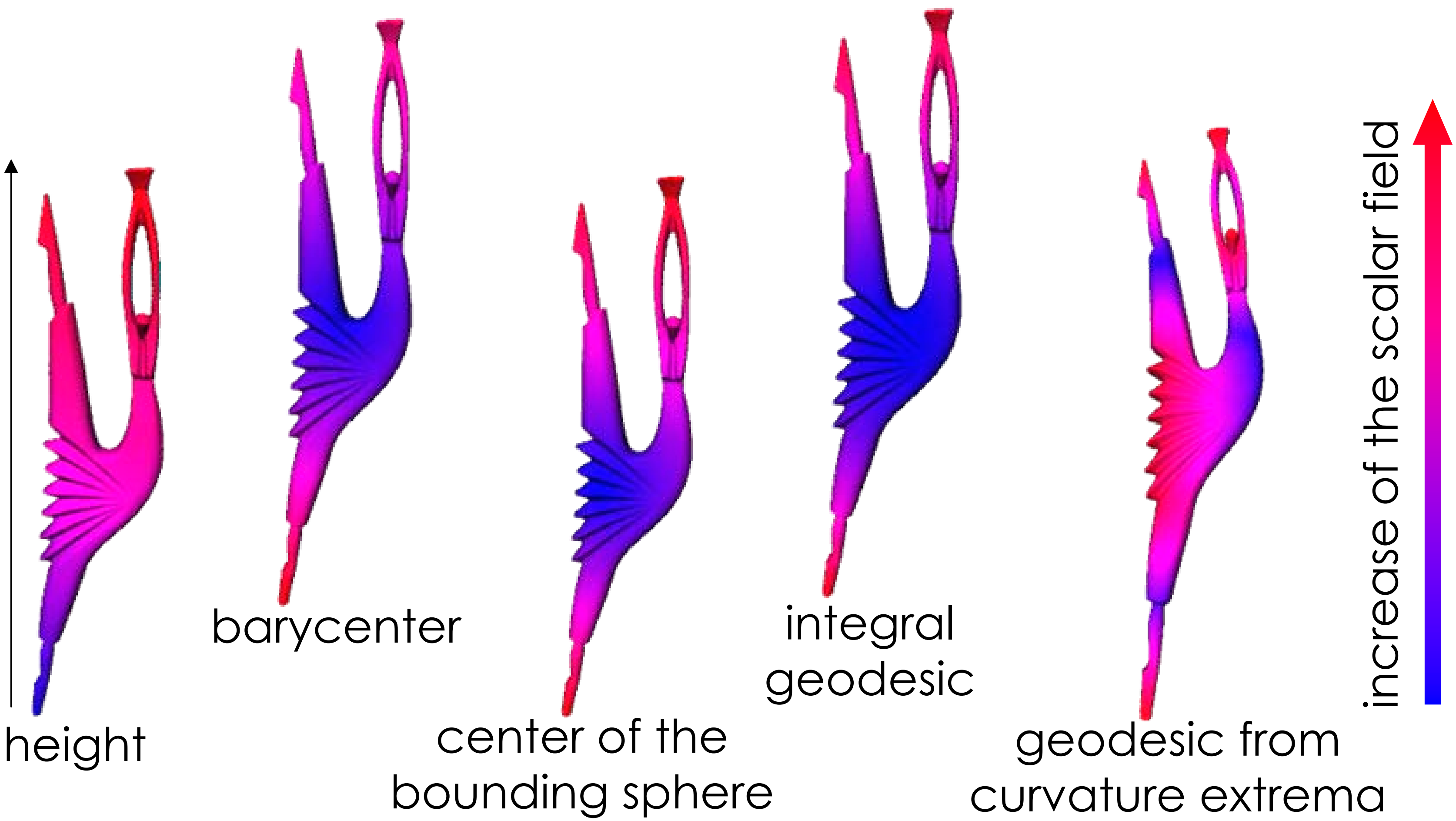

  - they extend the Euler relationship, eg:

$$\sum_v i(v,f)=\chi(S)=1+1+1-1=2$$

- on any smooth compact manifold there exist Morse functions
- Morse functions are everywhere dense in the space of all smooth functions on the manifold
- on a compact manifold, any Morse function has only a finite number of critical points
- the set *S* of all simple Morse functions is everywhere dense in the set of all Morse functions
- *height function, distance functions, geodesic distance, etc* are example of functions almost everywhere Morse

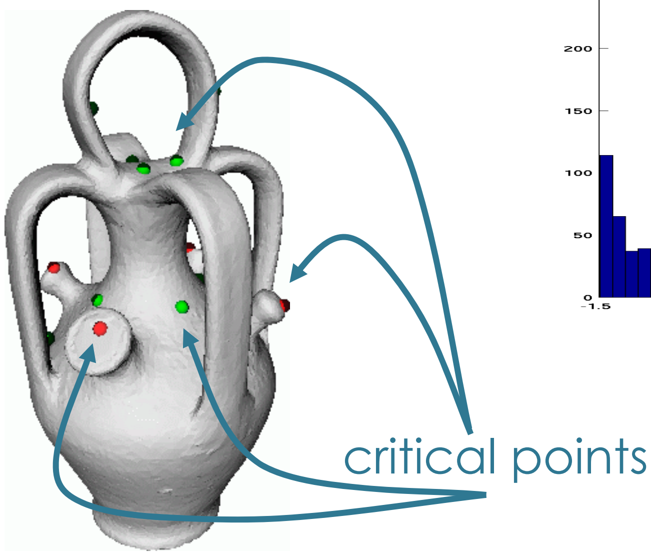
# shapes and real functions

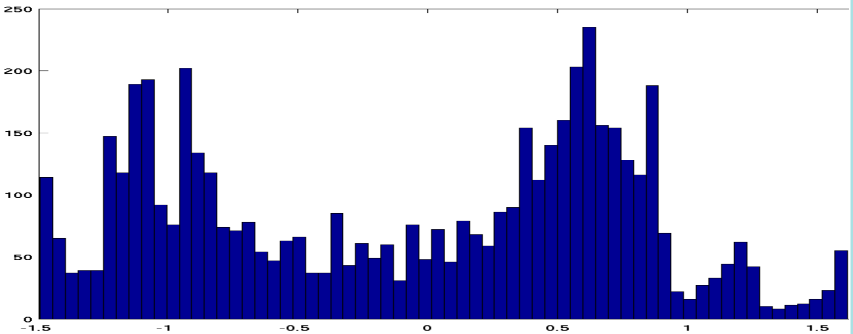
- ✓ different functions "measure" different characteristics of a shape



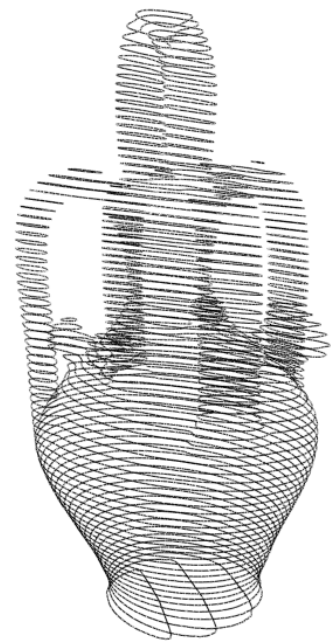
✓ analysis of

– the point-wise variation of *f*-values



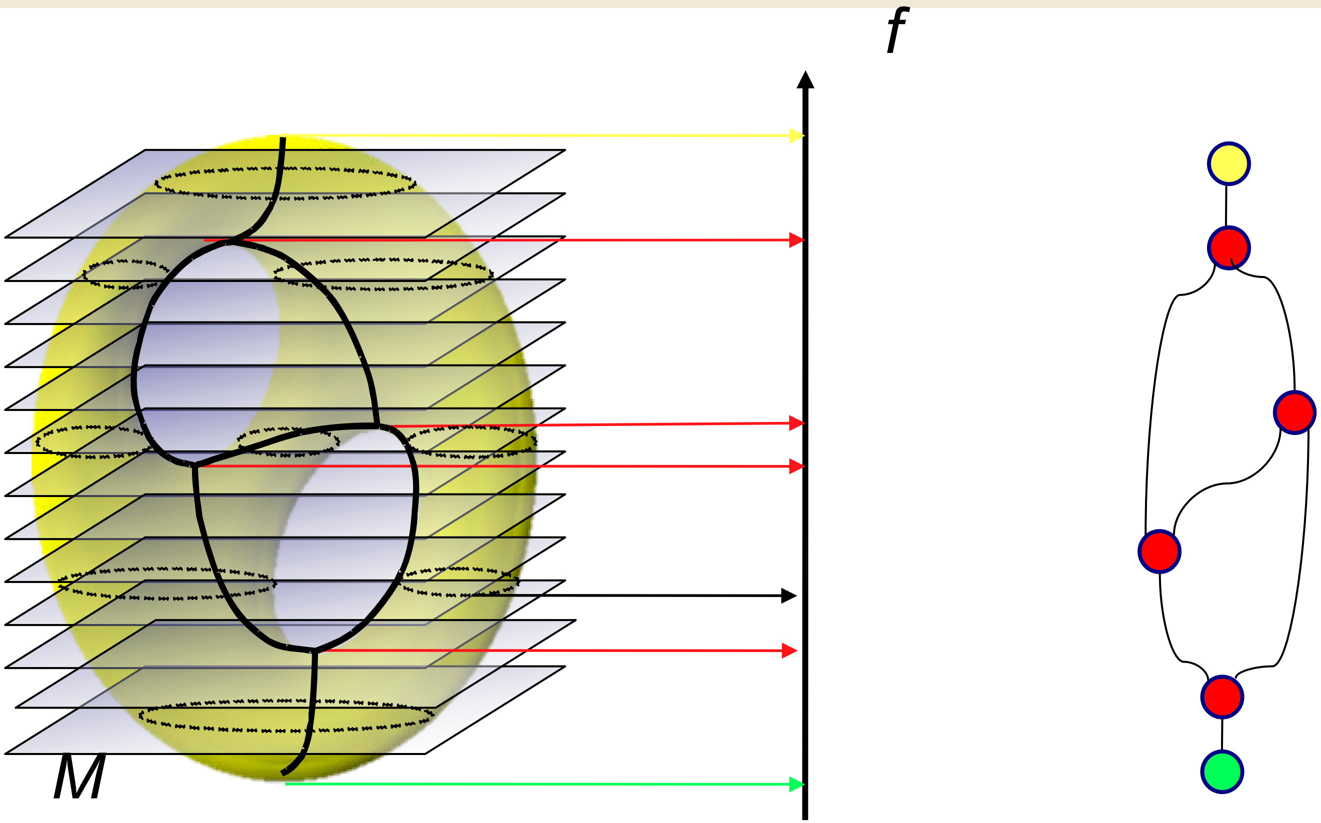

– the location of the critical points



– the evolution of the level sets

# shape descriptors: some examples

- ✓ medial axis transform (Blum 1967)
- ✓ shock graphs (Kimia,Tannenbaum, Zucker 1995)
- ✓ surface networks (Pfaltz 1976)
- ✓ Morse and Morse-Smale complexes (Edelsbrunner et al. 2001, Edelsbrunner et al. 2003)
- ✓ skeleton graphs and centerlines (Gagvani&Silver 1999, Lazarus&Verroust 1999, Dey et al 2003,2006)
- ✓ apparent contours
- ✓ persistence homology tools (Edelsbrunner et al 2002, Edelsbrunner&Harer to appear)
- ✓ size functions (Ferri, Frosini 1990)
- ✓ Reeb graphs (Reeb 1946)



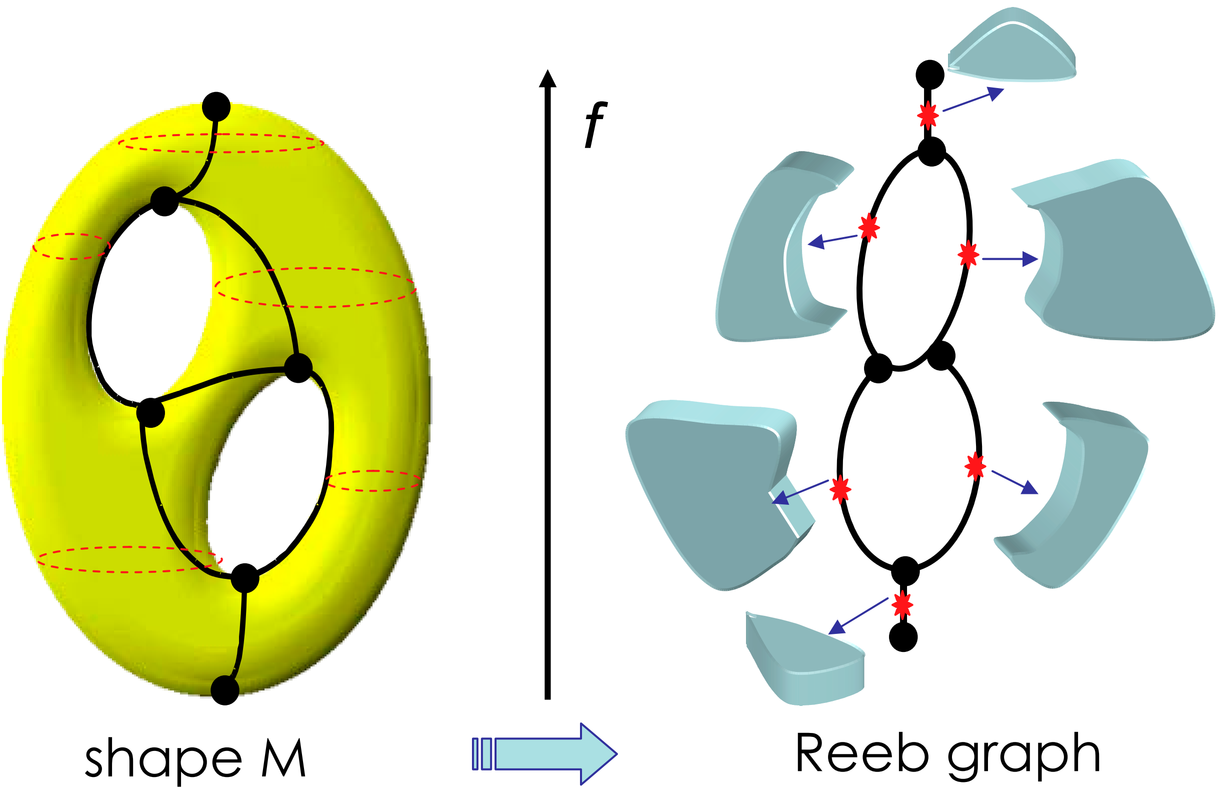

- Reeb graphs describe the shape of M by coding the evolution of the level sets of the mapping function f on M

# Reebgraph definition

- ✓ let M be a compact n-manifold and $f: M \rightarrow R$ a simple Morse function and let "~" be the equivalence relation:
- ✓ $(P, f(P)) \sim (Q, f(Q)) \Leftrightarrow f(P) = f(Q)$ and *P* and Q are in the same connected component of $f^{-1}(f(P))$
- ✓ the quotient space on *MxR* is a finite, connected simplicial complex *K* of dimension *1*, such that
  - the counter-image of each vertex of *K* is a singular connected component of the level sets of *f*
  - the counter-image of the interior of each simplex of dim *1* is homeomorphic to the topological product of one connected component of the level sets by *R*

G.Reeb. Sur le points singuliers d'une forme de Pfaff completement integrable ou d'une fonctionn numerique. Comptes Rendus Acad. Science, Paris, 1946, 222: 847-849

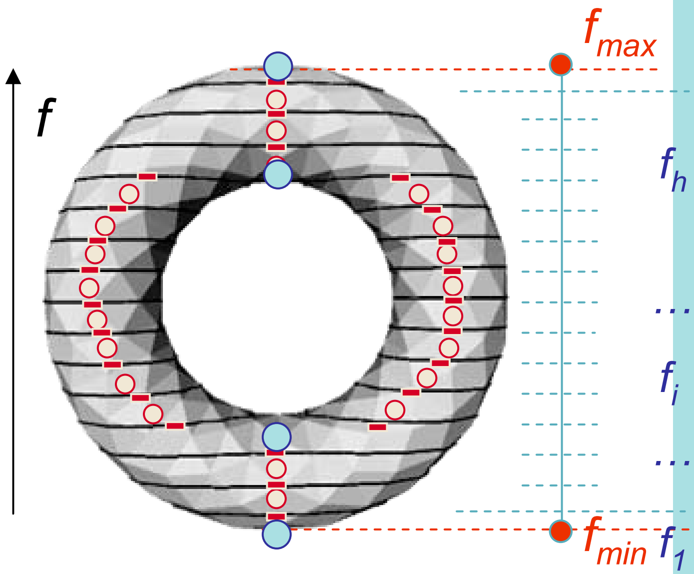
# Reeb graphs as shape descriptors




- ✓ the 1-simplex is often associated to a geometric embedding *(centerline skeleton)*, or used to store additional geometric data

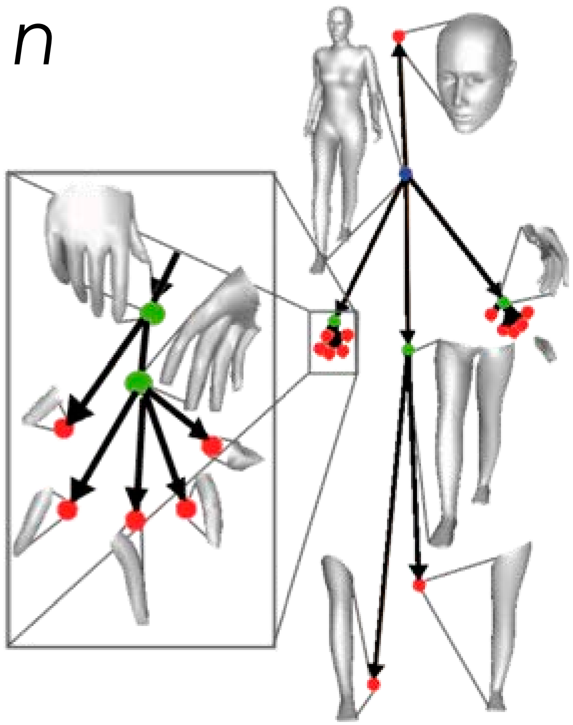
# our approach: Extended Reeb Graph (ERG)

- ✓ founds on an extended Reeb equivalence
  - let $f: M \rightarrow R$ be a continuous function;
  - let $I=\{(f_{min}, f_1),(f_h,f_{max}),(f_i,f_{i+1}),i=1\ldots h-1\} \cup \{f_{min},f_1,\ldots f_h, f_{max}\}$ be a partition of $[f_{min}, f_{max}]$;
  - the *extended Reeb equivalence* between $P, Q \in M$ is given by:
    - $f(P)$, $f(Q)$ belong to the same element of $I$;
    - $P$, $Q$ belong to the same connected component of $f^{-1}(f(t))$, $t \in I$.
- ✓ the ERG is a 1-simplex made of a finite number of elements



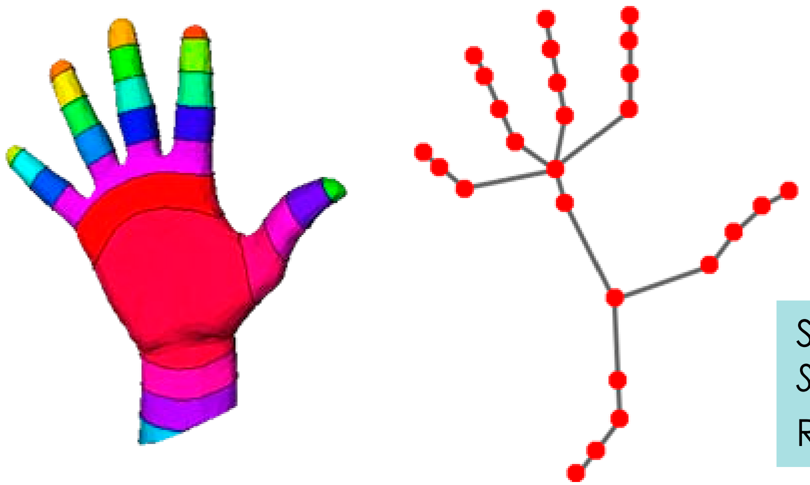

S. Biasotti, B. Falcidieno, M. Spagnuolo, Surface Shape Understanding based on Extended Reeb Graphs, *Surface Topological Data Structures*, pp. 87-103, John Wiley & Sons, 2004

- each arc can be oriented using the increasing direction of *f*: the ERG is a direct acyclic graph
- store with each node *n* a representation of the sub-graph associated with *n*
- for each arc e, compute the number of slices traversed by the arc (arc length)



S. Biasotti, S. Marini, M. Spagnuolo, B. Falcidieno *Sub-part Correspondence by Structural Descriptors of 3D Shapes*. CAD, 38 (9):1002–1019, 2006

- ✓ nodes of the ERG correspond to surface regions
- ✓ arcs code the adjacency among the regions
- ✓ node attributes code spatial properties (area, perimeter, radius, etc) of the region associated to that node



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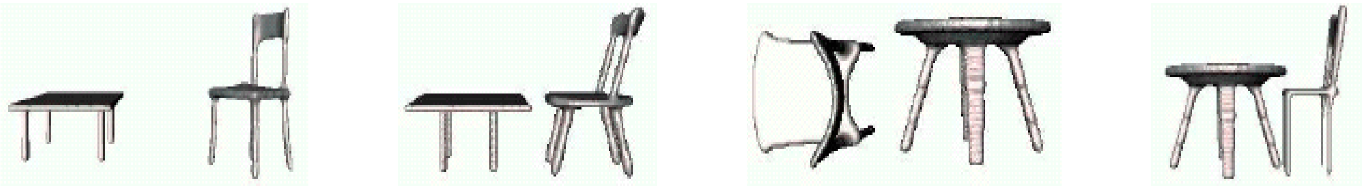
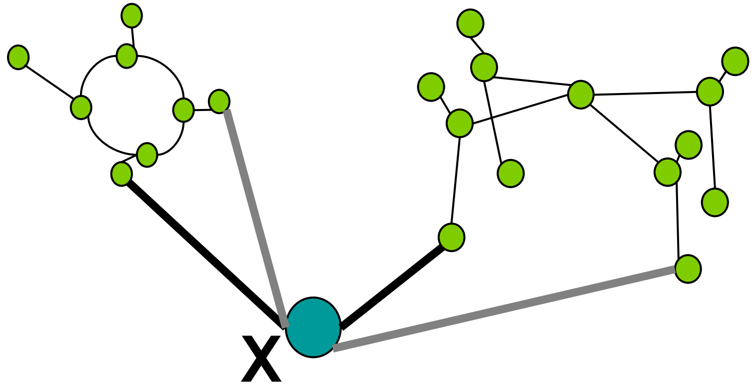


- a scene root X is added to the set of graphs



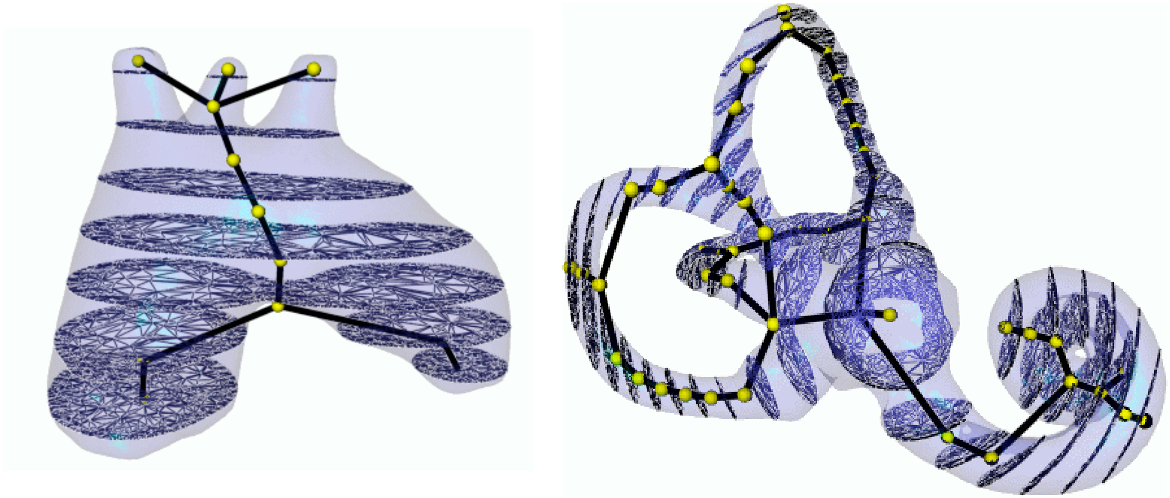

- the Laplacian spectrum $L = \Phi\Phi^t, \Phi = (\sqrt{\lambda_1}e_1,\ldots,\sqrt{\lambda_n}e_n)$ of the forest is used to define the feature vector B

$$B = (F_{11},\ldots,F_{1n},\ldots,F_{n1},\ldots,F_{nn})^t$$

where $F_{ij} = sign(S_j(\Phi_{1i},\ldots,\Phi_{ni}))\ln(1+|S_j(\Phi_{1i},\ldots,\Phi_{ni})|)$

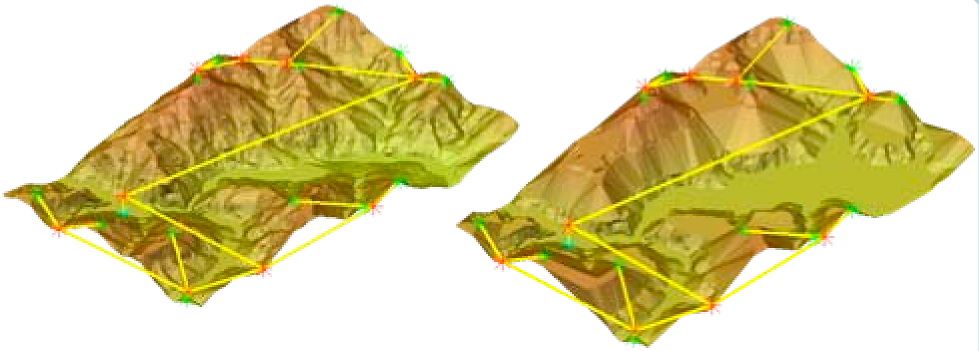
$$S_j(x_1,\ldots,x_n) = \sum_{i_1<\ldots<i_j} x_{i_1}\cdot\ldots\cdot x_{i_j}$$

- the ERG is extracted from a finite set of iso-surfaces

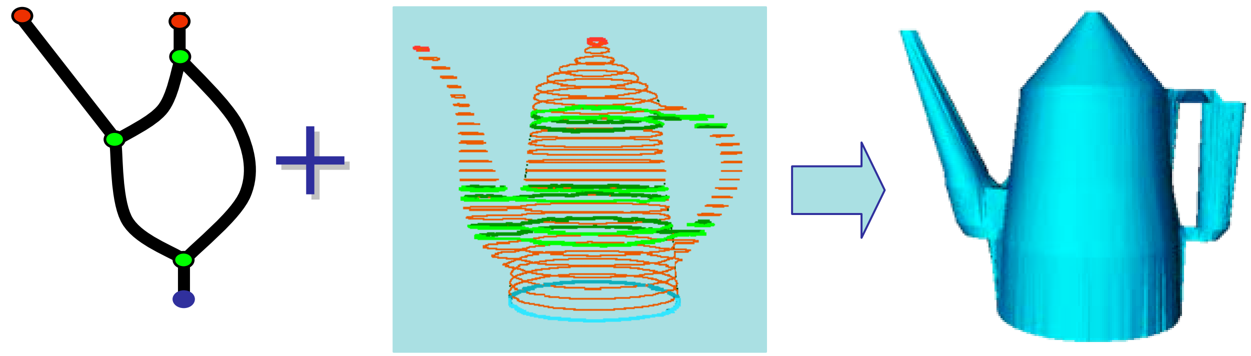


- we introduce the concept of critical volume
  - differently from surfaces, passing through a saddle, the genus of $f^{-1}(f(t))$ may change without varying its number connected components
- nodes and arcs are coupled with geometric attributes that describe the volume subparts

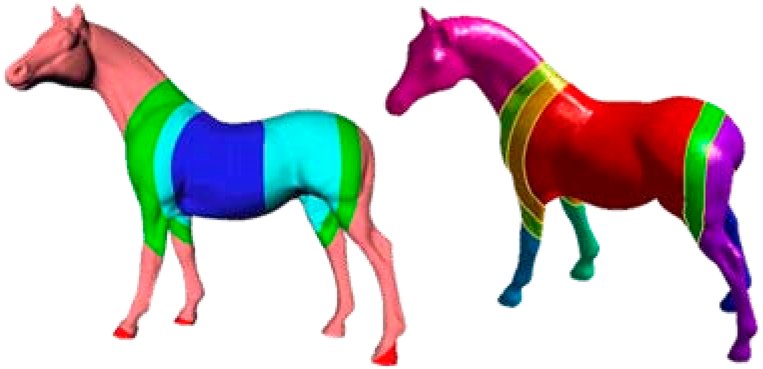
- topology simplification



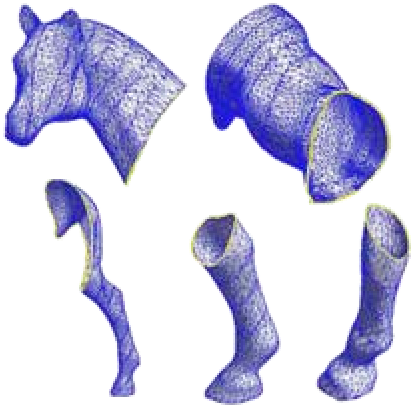
- topology simplification
- shape analysis and understanding



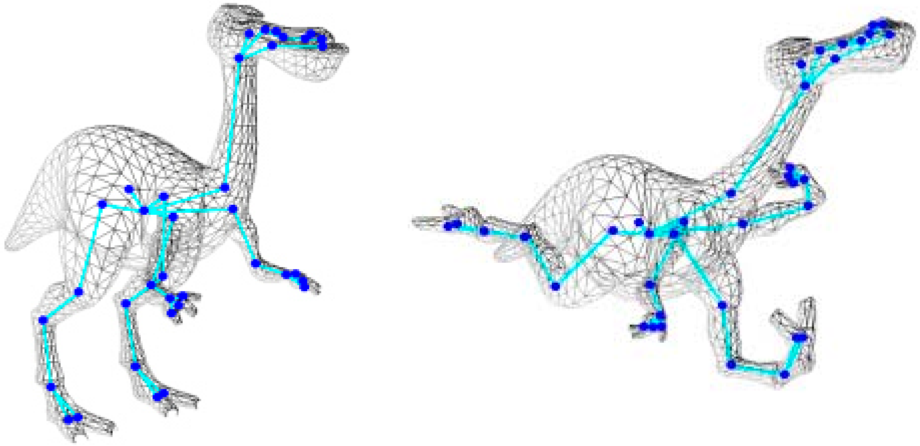
- topology simplification
- shape analysis and understanding
- shape and body segmentation



- topology simplification
- shape analysis and understanding
- shape and body segmentation
- shape parameterization

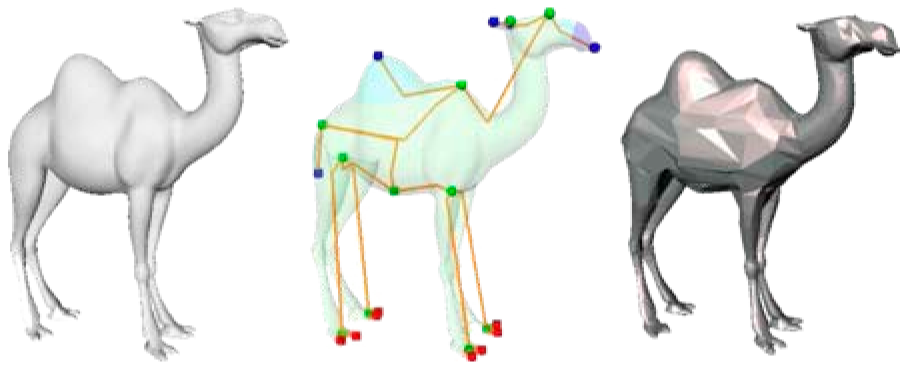


- topology simplification
- shape analysis and understanding
- shape and body segmentation
- shape parameterization
- handle identification
- animation

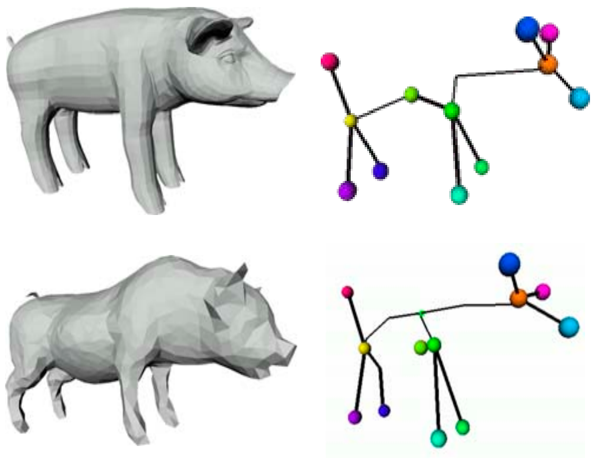


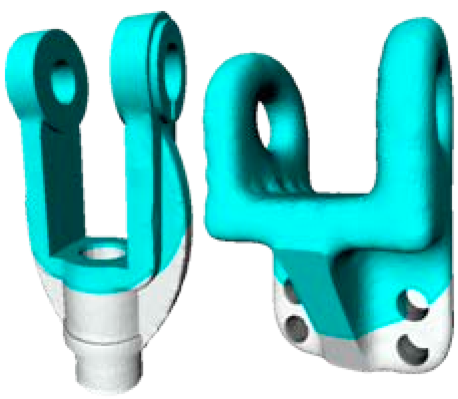
# applications

- ✓ topology simplification
- ✓ shape analysis and understanding
- ✓ shape and body segmentation
- ✓ shape parameterization
- ✓ handle identification
- ✓ animation
- ✓ shape coding and approximation



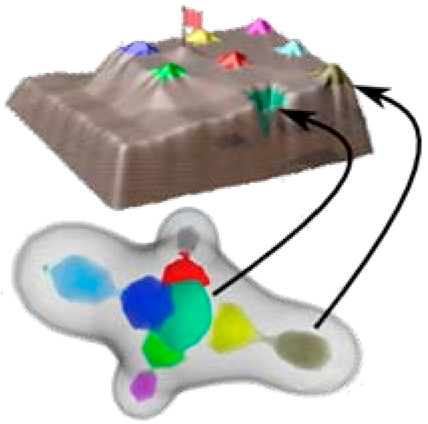
- topology simplification
- shape analysis and understanding
- shape and body segmentation
- shape parameterization
- handle identification
- animation
- shape coding and approximation
- global and partial matching





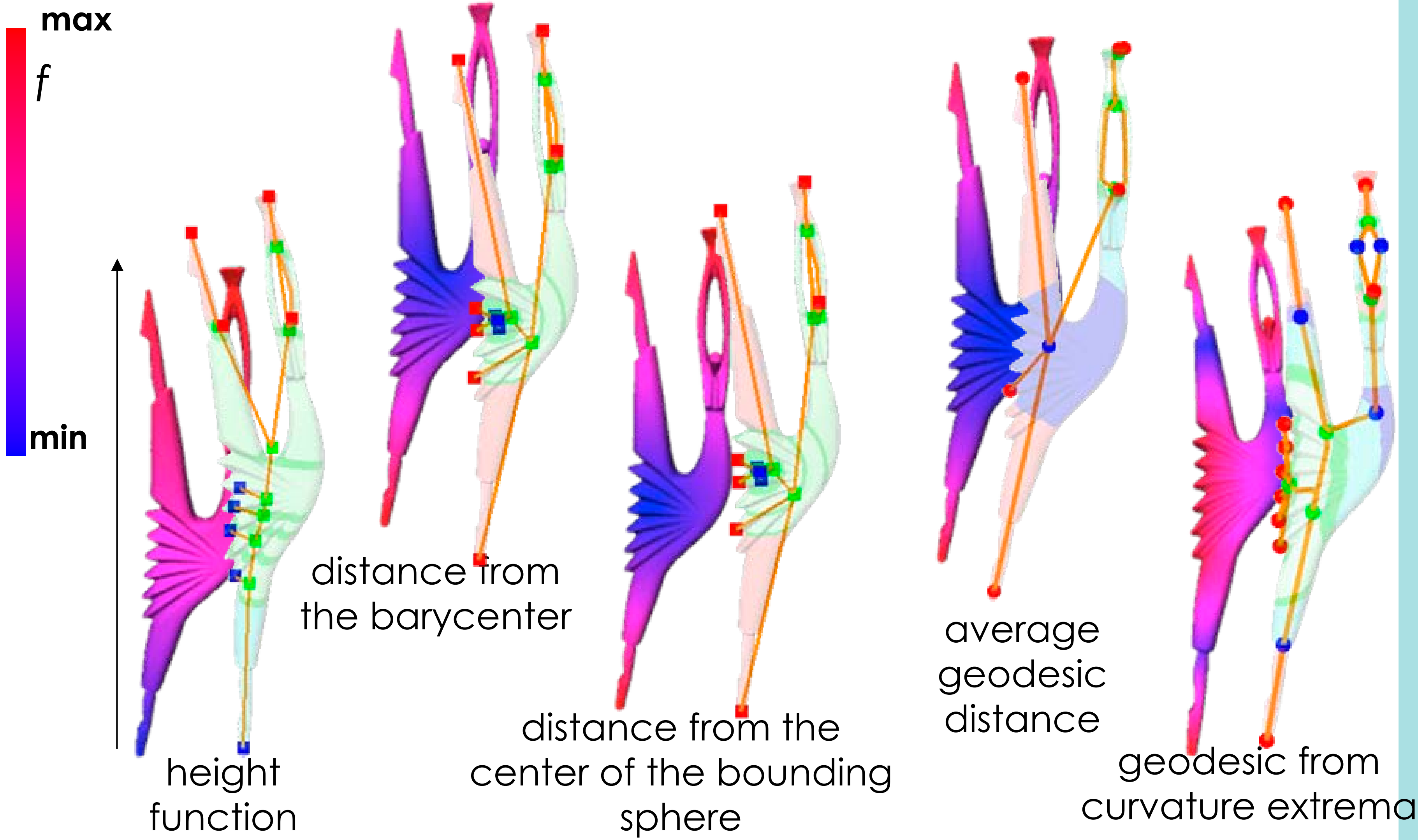
# applications

- ✓ topology simplification
- ✓ shape analysis and understanding
- ✓ shape and body segmentation
- ✓ shape parameterization
- ✓ handle identification
- ✓ animation
- ✓ shape coding and approximation
- ✓ global and partial matching
- ✓ shape classification
- ✓ volume visualization
- ✓ scientific visualization



- topology simplification
- shape analysis and understanding
- shape and body segmentation
- shape parameterization
- handle identification
- animation
- shape coding and approximation
- global and partial matching
- shape classification
- volume visualization
- scientific visualization
- rendering
- X-ray cristallography
- analysing of time-varying data

# Reeb graphs inherit the invariance properties of f



- ✓ choice of good functions
  - are there intrinsic properties?
  - what about Laplacian eigenfunctions?
  - use of statistical approaches (e.g., machine learning) or relevance feedback techniques

- Biasotti S., Cerri A., Frosini P., Giorgi D., Landi C. **Multidimensional size functions for shape comparison**, *Journal of Mathematical Imaging and Vision*, Springer, 32(2):161-179, 2008
- Biasotti S., Giorgi D., Spagnuolo M., Falcidieno B., **Reeb graphs for shape analysis and applications**, *Theoretical Computer Science*, Elsevier, 392(1-3):5-22, 2008
- Biasotti S., De Floriani L., Falcidieno B., Frosini P., Giorgi D., Landi C., Papaleo L., Spagnuolo M. **Describing shapes by geometrical-topological properties of real functions**, *ACM Computing Surveys*, 40(4):1-87, 2008
- Biasotti S., Giorgi D., Spagnuolo M., Falcidieno B., **Size functions for comparing 3D models**, *Pattern Recognition*, Elsevier, 41(9):2855-2873, 2008
- Patanè G., Spagnuolo M., Falcidieno B. **A minimal contouring approach to the computation of the Reeb graph,** *IEEE Transactions on Visualization and Computer Graphics*. 2009.
- Reuter M., Biasotti S., Giorgi D., Patanè G., Spagnuolo M. **Discrete Laplace-Beltrami Operators for Shape Analysis and Segmentation**, *Computer&Graphics*. Elsevier, 33(3), xx 2009.
- Biasotti S., Patanè G., Spagnuolo M., Falcidieno B., Barequet G. **Shape approximation based on differential properties,** *Computer&Graphics*. Elsevier, 34(3), xx 2010.

# acknowledgments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- CNR Project *ICT.P10.007*: Codifica, elaborazione e restituzione della conoscenza legata a media multidimensionali

- extension of the representation framework to high dimensional and time dependent data
- to simplify shape retrieval from large datasets: proof of the stability of the Extended Reeb graph representation with respect to the graph editing function *d*
- definition of a theoretical paradigm for visual shape analysis and synthesis purely based on mathematical properties