Spectral Shape Analysis: Numerical Methods and Applications

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Outline

- **Feature spaces**
  - definitions & desiderata
- **Laplacian maps**
  - Laplacian eigenfunctions
  - harmonic functions
  - regularized and geometry-aware basis
  - heat kernel
- **Applications & Generalizations**
  - functions&surface smoothing/approximation
  - shape comparison
  - ...

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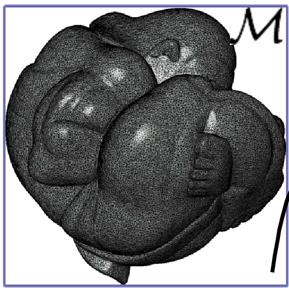
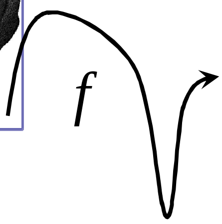
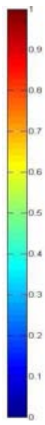
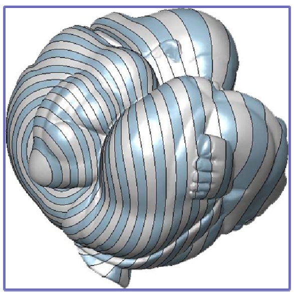
Feature space



$$\mathcal{F}(\mathcal{M}) := \{f : \mathcal{M} \to \mathbb{R}, f \text{ PL}\}$$



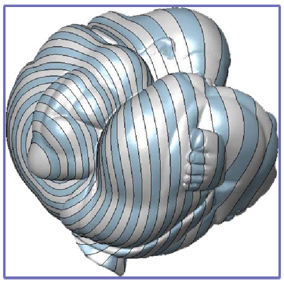




$$\mathcal{F}(\mathcal{M}) \approx \mathbb{R}^n$$

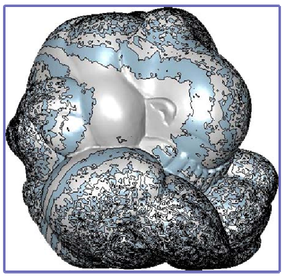
$(\mathcal{M}, f)$

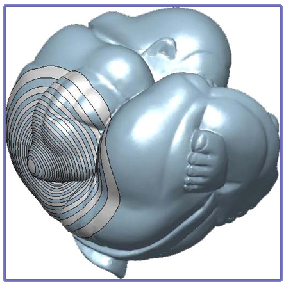
Feature space

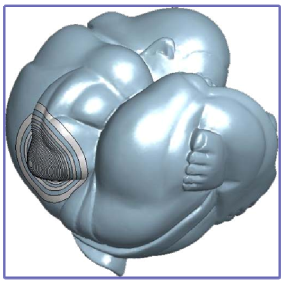


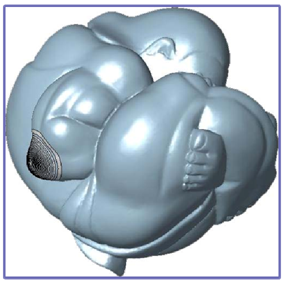
Global support

noise









Similar local behavior, compact support

Feature space

– How to
  - generate functions in F(M);
  - design functions with specific differential properties and behavior;
  - select functions in F(M);
  - measure properties/distances in F(M);
  - address applications: function/shape smoothing, approximation, compression, comparison, deformation;
  - …

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Laplace-Beltrami operator

$\mathcal{N}$ manifold

$$\begin{array}{cccc} \Delta : & \mathcal{C}^2(\mathcal{N}) & \longrightarrow & \mathcal{C}^0(\mathcal{N}) \\ & f & \mapsto & \Delta f \end{array}$$

Laplacian eigenfunction

$$\Delta f = \lambda f$$

Harmonic function

$$\begin{cases} \Delta f = 0 \\ f|_{\mathcal{B}} := f_0 \end{cases}$$

Heat equation

$$\partial_t f(\mathbf{x}, t) = -\frac{1}{2}\Delta f(\mathbf{x}, t)$$

Laplacian matrix

Consider the **linear FEM discretization** of the Laplace Beltrami operator

$$\Delta \approx \underbrace{\tilde{L} := B^{-1}L}_{\in \mathbb{R}^{n\times n}}$$

Laplacian matrix

[Reuter06,VALLET09
]

$$f \in \mathcal{F}(\mathcal{M}) \leftrightarrow (f(\mathbf{p}_i))_{i=1}^{n} \in \mathbb{R}^n$$

$$\begin{array}{llll} \tilde{L}: & \mathcal{F}(\mathcal{M}) & \rightarrow & \mathcal{F}(\mathcal{M}) \\ & \mathbf{f} & \mapsto & \tilde{L}\mathbf{f} \end{array}$$

Feature map

Laplacian matrix

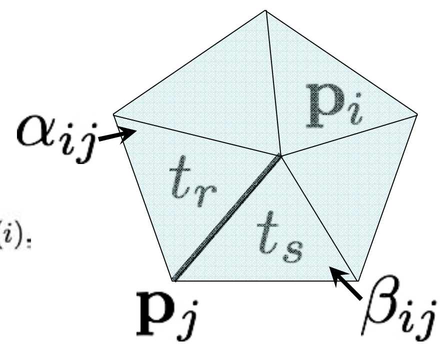
Stiffness matrix L

$$\tilde{L} := B^{-1}L$$

$$L(i,j) := \begin{cases} w(i,j) := \frac{\cot\alpha_{ij} + \cot\beta_{ij}}{2} & j \in N(i), \\ -\sum_{k\in N(i)} w(i,k) & i = j, \\ 0 & \text{else}, \end{cases}$$

Mass matrix B

$$B(i,j) := \begin{cases} \frac{|t_r|+|t_s|}{12} & j \in N(i), \\ \frac{\sum_{k\in N(i)} |t_k|}{6} & i = j, \\ 0 & \text{else}, \end{cases}$$



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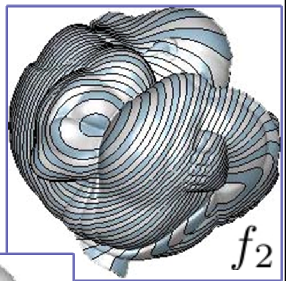
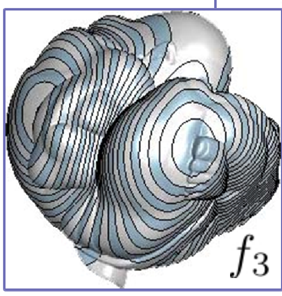
Laplacian spectrum [NGH04,RWP06]

- Consider the spectrum of the Laplacian matrix associated to *M*

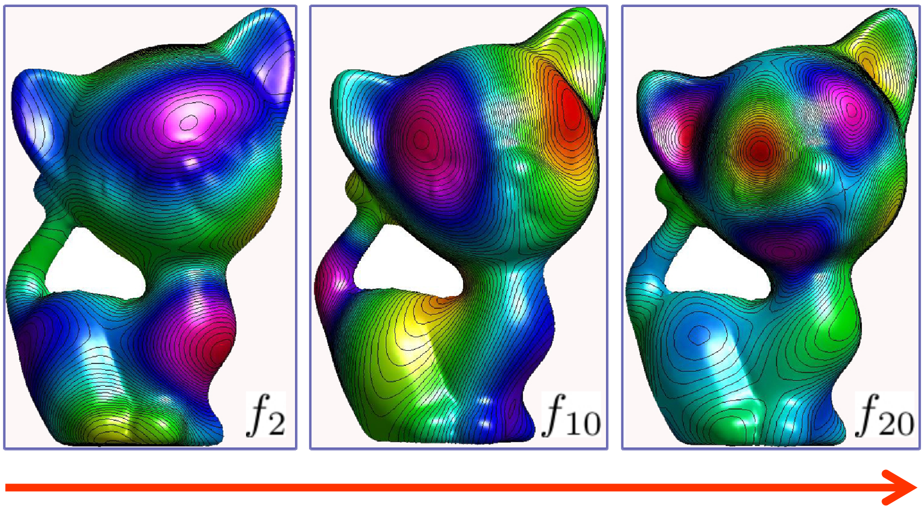
$$L\mathbf{x}_i = \lambda_i B\mathbf{x}_i,\ i = 1, \dots, n$$
$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$



$$f_i : \begin{array}{ccc} \mathcal{M} & \rightarrow & \mathbb{R} \\ \mathbf{p}_k & \mapsto & \mathbf{x}_i^{(k)} \end{array}$$



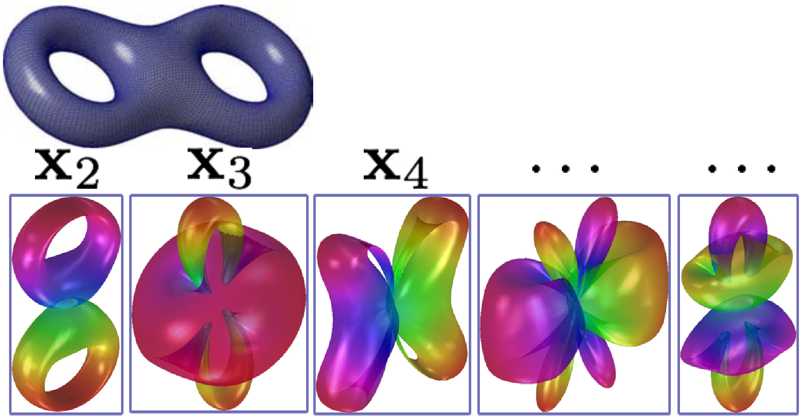
Laplacian spectrum



Higher eigenvalue→higher complexity in term of level-sets & critical points

Laplacian spectrum

– Analogies with the spherical harmonics.



$$\langle \mathbf{x}_i, \mathbf{x}_j \rangle_{\mathcal{F}(\mathcal{M})} = \delta_{ij} \qquad \mathbf{f} = \sum_{i=1}^{n} \langle \mathbf{f}, \mathbf{x}_i \rangle_{\mathcal{F}(\mathcal{M})} \mathbf{x}_i$$

Laplacian spectrum

- The spectrum defines a set of functions, which are
  - "intrinsically" defined by the input shape;
  - smooth functions in terms of level-sets and critical points;
  - provide a basis of the feature space.

Reuter M., Biasotti S., Giorgi D., Patanè G, Spagnuolo M. *Discrete Laplace-Beltrami Operators for Shape Analysis and Segmentation.* In: Computers & Graphics, vol. 33 (3) pp. 381 - 390. Elsevier, 2009.

Ruggeri M. R., Patanè G., Spagnuolo M., Saupe D. *Spectral-driven isometry-invariant matching of 3D shapes.* In: International Journal of Computer Vision. May 2009.

Laplacian spectrum

- The spectrum defines a set of functions, which are
  - "intrinsically" defined by the input shape;
  - smooth functions in terms of level-sets and critical points;
  - provide a basis of the feature space.

- No control over the
  - function support;
  - alignment of the function with shape features;
  - computation of the whole spectrum;
  - eigenfunctions' switch.

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Harmonic functions [DBG*06]

Smooth functions with a (generally) low number of critical points are achieved by solving the Laplace equation with Dirichlet boundary conditions.

$$\begin{cases} \Delta f = 0 \\ f(\mathbf{p}_i) = \alpha_i, \quad i \in \mathcal{I} \end{cases}$$

Laplace equation

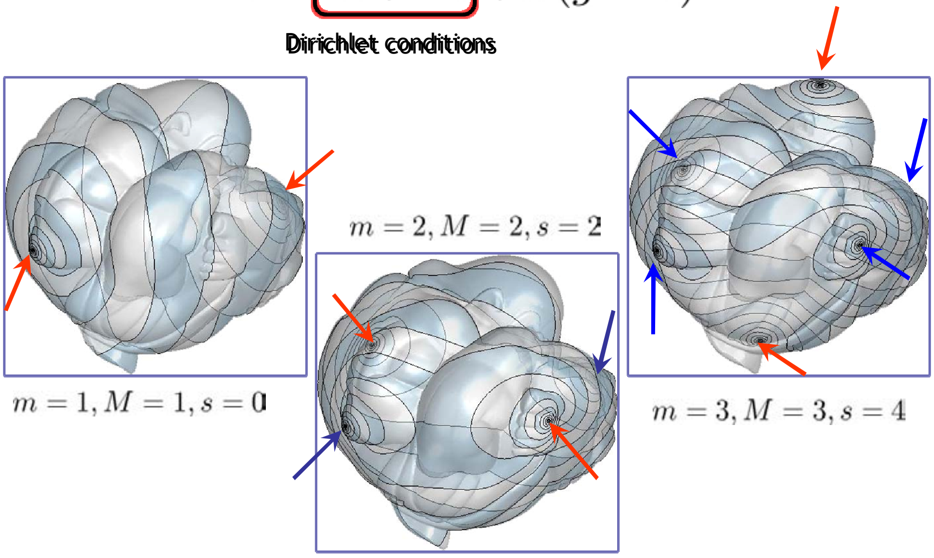
Dirichlet boundary conditions

$\mathcal{I} \subseteq \{1, \ldots, n\}$

Harmonic functions

$$s = \boxed{m + M} + 2(g - 1)$$

Dirichlet conditions



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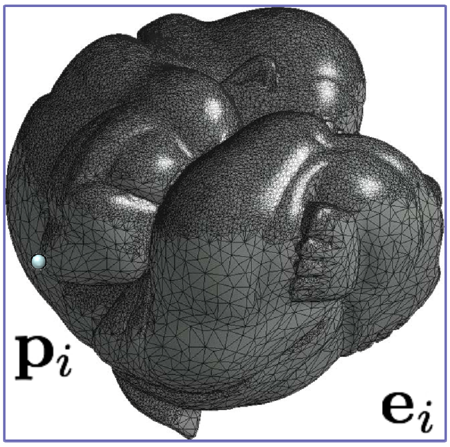
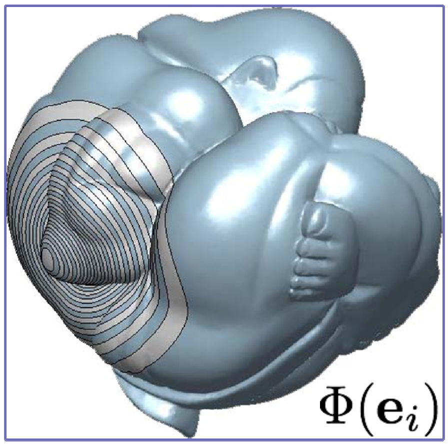
Smoothing & Geometry-aware functions

$$e_i : \mathcal{M} \to \mathbb{R}$$

$$e_i(\mathbf{p}_j) = \delta_{ij}$$

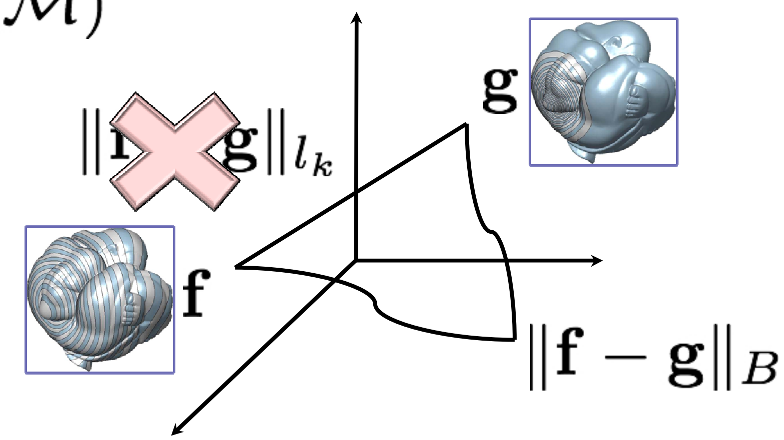


$$\Phi : \mathcal{F}(\mathcal{M}) \longrightarrow \mathcal{F}(\mathcal{M})$$



Metrics in the feature space

$\mathcal{F}(\mathcal{M})$



B pos. definite $\langle \mathbf{f}, \mathbf{g} \rangle_B := \mathbf{f}^T B \mathbf{g}$

Tikhonov regularization

$f \in \mathcal{F}(\mathcal{M})$ **Input (noisy) map**

$(\text{e.g., } f = e_i)$

Compromise between approximation accuracy and smoothness

$$\mathcal{F}(\tilde{\mathbf{f}}) := \boxed{\epsilon}\,\boxed{\|\tilde{\mathbf{f}} - \mathbf{f}\|_B^2}\,\boxed{+\,\|L\tilde{\mathbf{f}}\|_2^2}$$

Approximation accuracy is measured with respect to the norm induced by a positive definite matrix B (e.g., B:=I or linear FEM mass matrix)

Smoothness is controlled by the Laplacian matrix M

Tikhonov regularization

– Minimize the **quadratic functional**

$$\mathcal{F}(\tilde{\mathbf{f}}) := \epsilon\|\tilde{\mathbf{f}} - \mathbf{f}\|_B^2 + \|L\tilde{\mathbf{f}}\|_2^2$$

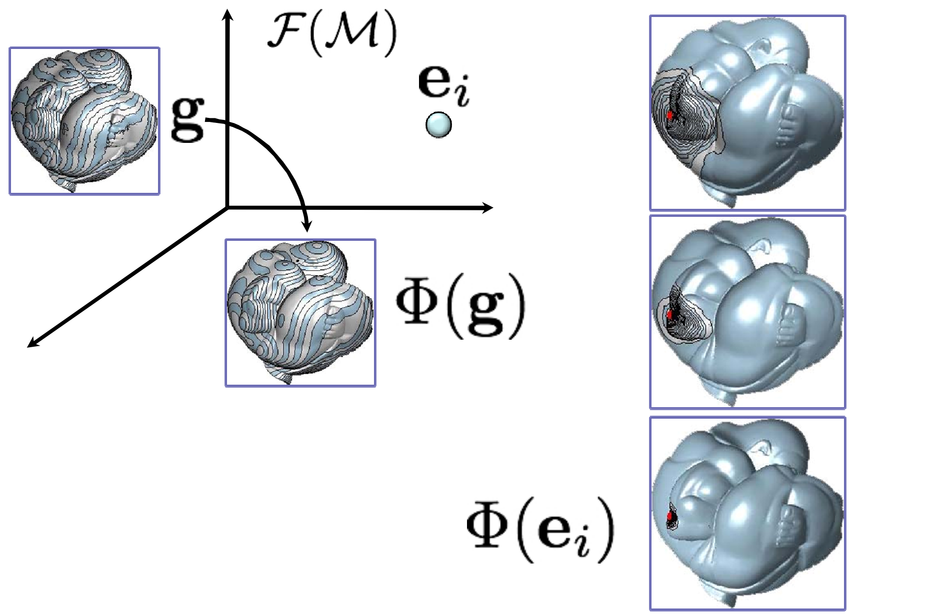
– equivalent to compute the solution of the **normal equation**

$$(L^T L + \epsilon B)\tilde{\mathbf{f}} = \epsilon B\mathbf{f}.$$

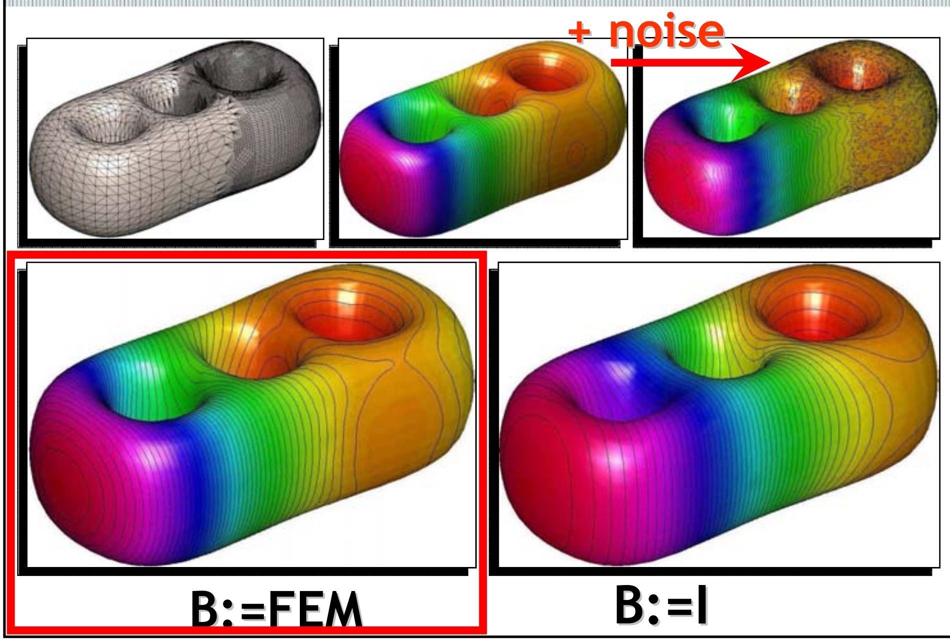
– The **coefficient matrix** is
  - symmetric;
  - positive definite;
  - sparse (as sum of two sparse matrices).

Patanè G, Falcidieno B. *Computing Smooth Approximations of Scalar Functions with Constraints*. In: Computers & Graphics, vol. 33 (3) pp. 399 - 413. Elsevier, 2009.

Smoothing & Geometry-aware functions



Robustness w.r.t. sampling



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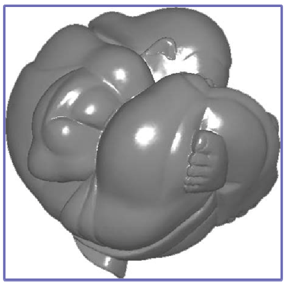
Heat kernel

$\mathcal{N}$ manifold



$$H : \mathcal{N} \times \mathbb{R} \rightarrow \mathbb{R}$$

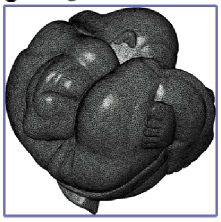
$$\begin{cases} \partial_t H(\mathbf{x}, t) = -\frac{1}{2}\Delta H(\mathbf{x}, t), \\ H(\mathbf{x}, 0) = h(\mathbf{x}), \end{cases}$$

$\{(\lambda_i, \phi_i)\}_{i=1}^{n}$ Eigensystem of Δ

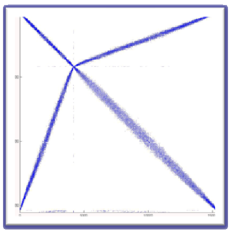
$$H(\mathbf{x}, t) = k_t(\mathbf{x}, \cdot) \star h \quad k_t(\mathbf{x}, \mathbf{y}) := \sum_{i=0}^{+\infty} \exp(-\frac{1}{2}\lambda_i t)\phi_i(\mathbf{x})\phi_i(\mathbf{y})$$

Discrete heat kernel: B:=I

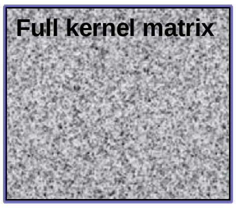
$\mathcal{M}$



$\Delta \approx L$



$\{(\lambda_i, \mathbf{x}_i)\}_{i=1}^{n}$



$X := [\mathbf{x}_1, \dots, \mathbf{x}_n]$

$D_t := \mathrm{diag}(\exp(-\frac{1}{2}\lambda_i t))_{i=1}^{n}$

$K_t := X D_t X^T$

wFEM discrete heat kernel

– Use the linear FEM discretization of the Laplace-Beltrami operator (Galerkin formulation).

$$\begin{cases} \partial_t F(\mathbf{p}, t) = -\frac{1}{2}\tilde{L}F(\mathbf{p}, t), & \mathbf{p} \in \mathcal{M}, \\ F(\mathbf{p}_i, 0) = f(\mathbf{p}_i), & i = 1, \dots, n, \end{cases}$$

$$\tilde{L} := B^{-1}L \qquad (L\mathbf{x}_i = \lambda_i B\mathbf{x}_i)$$

$$F(\cdot, t) = \sum_{i=1}^{n} \exp\left(-\frac{1}{2}\lambda_i t\right) \langle \mathbf{f}, \mathbf{x}_i \rangle_B \mathbf{x}_i$$

Patanè G, Falcidieno B. *Multi-Scale Feature Spaces for Shape Processing and Analysis.* In: IEEE International Conference on Shape Modeling 2010. To appear.

wFEM heat kernel

$$K_t = XD_tX^TB$$

– is not symmetric but **self-adjoint** with respect to the norm induced by B; i.e.,

$$\langle \mathbf{f}, K_t\mathbf{g}\rangle_B = \langle K_t\mathbf{f}, \mathbf{g}\rangle_B$$

wFEM heat kernel

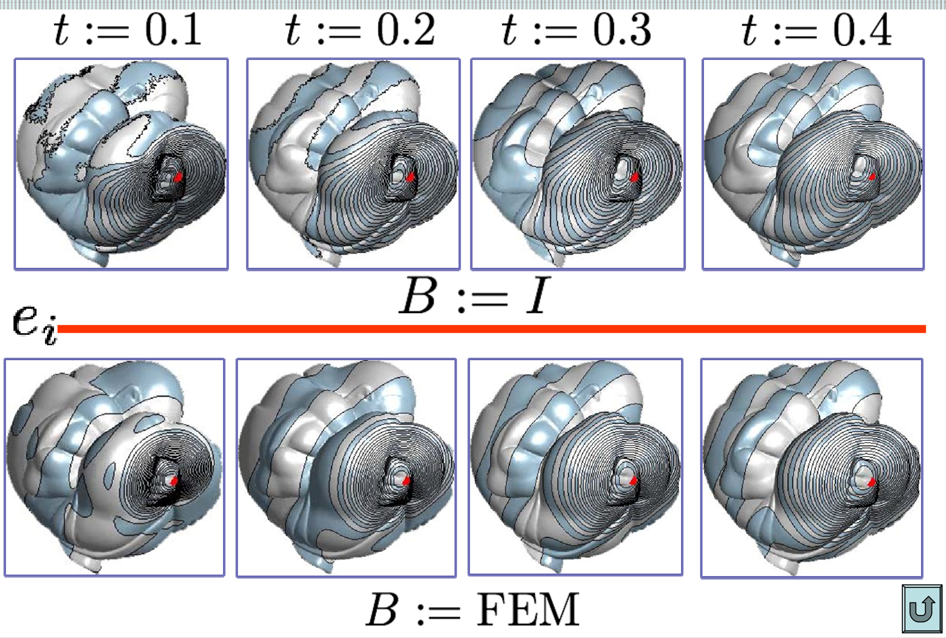
$$K_t = XD_tX^TB$$

– is not symmetric but self-adjoint with respect to the norm induced by B; i.e.,

$$\langle \mathbf{f}, K_t\mathbf{g}\rangle_B = \langle K_t\mathbf{f}, \mathbf{g}\rangle_B$$

– more robust w.r.t. irregular sampling densities;

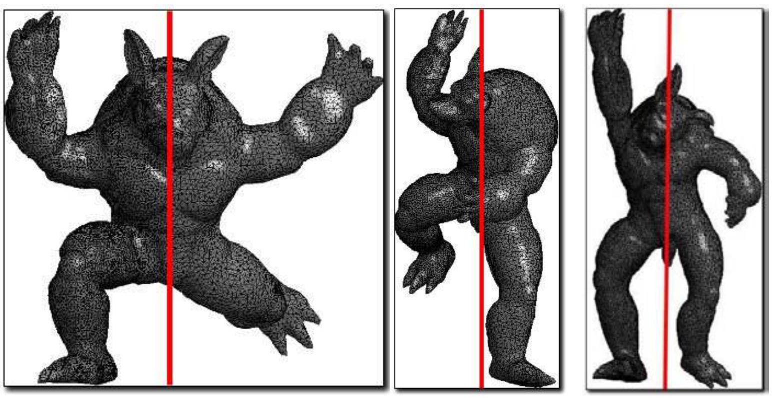
wFEM heat kernel: robustness



wFEM heat kernel

$$K_t = X D_t X^T B$$

– invariant w.r.t. isometric transformations;



wFEM heat kernel

Standard approach | **Proposed approach**

$$\partial_t H(\mathbf{p},t) = -\frac{1}{2}\Delta H(\mathbf{p},t)$$

Standard approach:

$$\partial_t F(\mathbf{p},t) = -\frac{1}{2} L F(\mathbf{p},t)$$

$$K_t := X D_t X^T$$

Proposed approach:

$$\partial_t F(\mathbf{p},t) = -\frac{1}{2} \underbrace{B^{-1} L}_{=:\tilde{L}} F(\mathbf{p},t)$$

$$K_t := X D_t X^T B$$

$B := I$

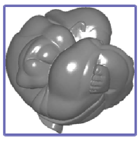
Galerkin formulation

wFEM heat kernel

$$K_t = X D_t X^T B$$

– is **scale-covariant** and outperforms previous work on heat kernel shape signatures.

$K_t(\mathcal{M})$



$\mathcal{M}$

Global/local shape rescaling

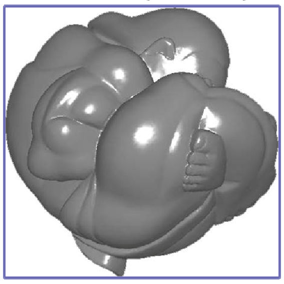
$$K_t(\alpha\mathcal{M}) = \boxed{\alpha^2} K_{\boxed{\alpha^2 t}}(\mathcal{M})$$

$$B := I$$

$$K_t(\alpha\mathcal{M}) = K_{\boxed{t/\alpha^2}}(\mathcal{M})$$

$$B \neq I$$

$K_t(\alpha\mathcal{M})$



$\alpha\mathcal{M}$

wFEM heat kernel

– How to guarantee that

$$K_t(\alpha\mathcal{M}) = K_t(\mathcal{M})$$

(i.e., **local&global scale invariance**).

$$K_t = X D_t X^T B$$

$$D_t \leftarrow \cancel{\exp\left(-\frac{1}{2}\lambda_i t\right)} \quad \exp\left(-\frac{1}{2}\frac{\lambda_i}{\lambda_2} t\right)$$

Bronstein A. M., Bronstein M. M., Bustos B., Castellani U., Crisani M., Falcidieno B., Guibas L. J., Kokkinos I., Murino V., Isipiran I., Ovsjanikov M., Patanè G., Spagnuolo M., Sun J. *SHREC 2010: robust large-scale shape retrieval benchmark.* Eurographics Workshop on 3D Object Retrieval. To appear.

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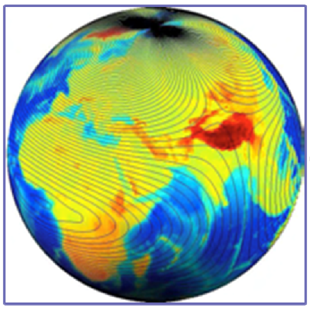
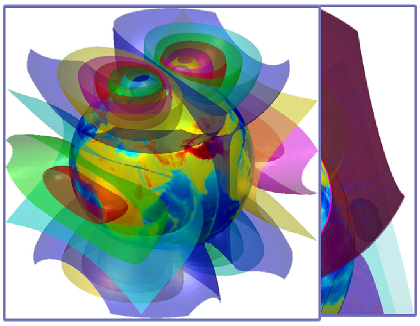
Generalizations

– From surfaces to volumes through topology-driven approximation of scalar functions with RBF functions.

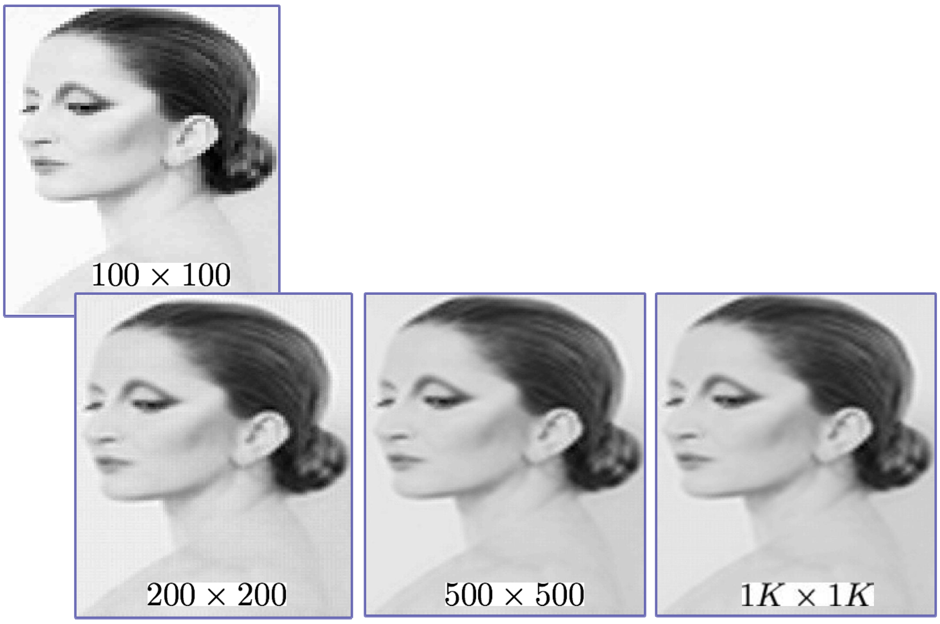
$f : \mathcal{M} \to \mathbb{R}$ $\qquad g|_{\mathcal{M}} \approx f$ $\qquad g : \mathbb{R}^3 \to \mathbb{R}$





Patanè G, Spagnuolo M., Falcidieno B. *Topology- and error-driven extension of scalar functions from surfaces to volumes.* In: ACM Transactions on Graphics. Volume 29, Issue 1, 2010. *To be presented at SIGGRAPH* 2010, *Los Angeles – USA, July* 25 – 29, 2010.

Multi-dimensional data approx.



Contact info & Acknowledgements

- **Additional info @**
  - Home page(s)
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    - http://www.ima.ge.cnr.it/ima/personal/patane/PersonalPage/Patanes_Home_Page/Home.html
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