3D Shape Description and Matching Based on Properties of Real Functions

Real functions

Eurographics 2007 Tutorial T12

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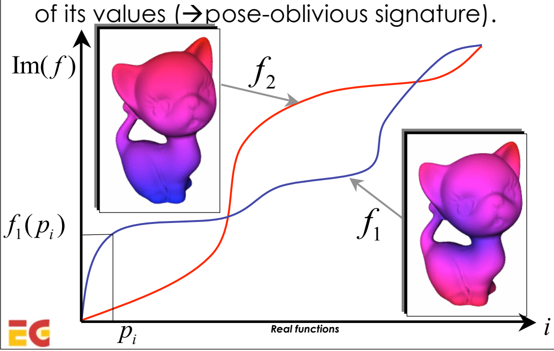
Outline

- Real functions on smooth/discrete surfaces:
  - General considerations
  - Differential and combinatorial properties
    - critical points
    - Euler formula.
  - Definitions
    - Height and elevation
    - Euclidean/geodesic distance function
    - Curvature-based functions
    - Local diameter
    - Harmonic functions and Laplacian eigenfunctions.
  - Properties
    - saliency, smoothness, stability
    - robustness, degrees of freedom and heuristics
    - efficiency, invariance.

Real functions 2

How can we study the behavior of functions on M?

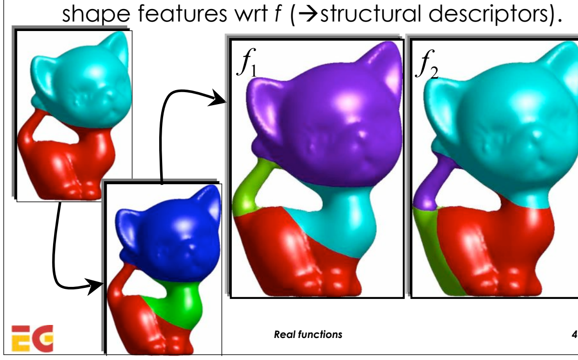
- Point-wise variation and statistical distribution of its values (→pose-oblivious signature).



Real functions 3

How can we study the behavior of functions on M?

- Distribution and hierarchical organization of shape features wrt *f* (→structural descriptors).



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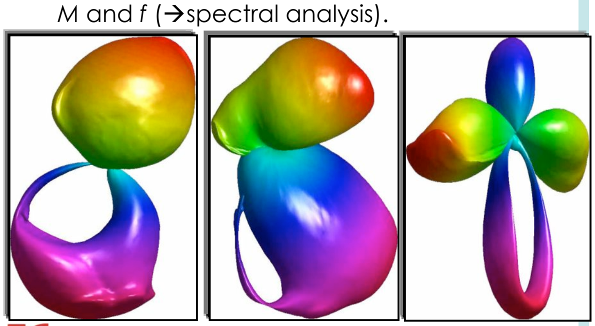
How can we study the behavior of functions on M?

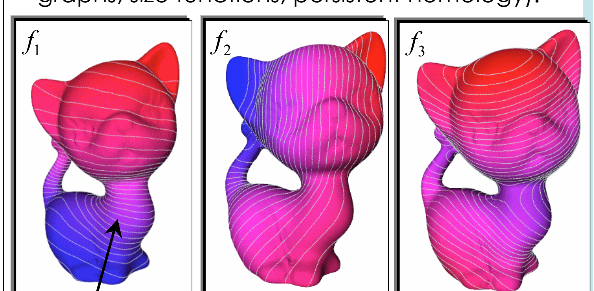
- Embedding in frequency spaces defined by M and *f* (→spectral analysis).



Real functions 5

How can we study the behavior of functions on M?

- Evolution of its (lower) level sets (→Reeb graphs, size functions, persistent homology).



$f^{-1}(\alpha)=\{p\in M : f(p)=\alpha\}$

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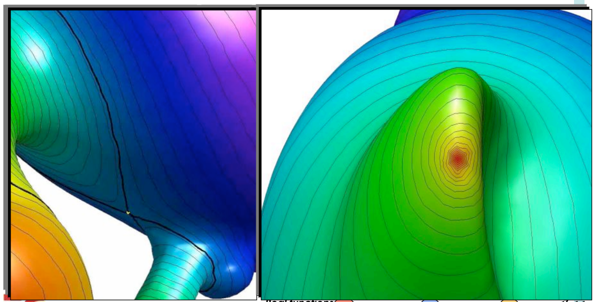
How can we study the behavior of functions on M?

- ✓ Number, type, and locations of its critical points (eg, Reeb graph).



Critical points [GP76,Mil63]

- ✓ Given a smooth function *f* defined on a manifold *M*:
  - a point x is called ***critical*** is the differential df_x is the zero map, that is,

$$\nabla f(x) = 0 \Leftrightarrow \frac{\partial f}{\partial x_1}(x) = 0, \frac{\partial f}{\partial x_2}(x) = 0, \ldots, \frac{\partial f}{\partial x_k}(x) = 0$$

  - a point x is called ***regular*** if the differential df_x is surjective, that is,

$$\nabla f(x) \neq 0$$

Critical points

- a critical point x is called ***non-degenerate*** if the Hessian matrix *H* of *f* is non-singular at *x*; then, *f* is called ***Morse*** at a x

$$|H_f(x)| = \left|\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right| \neq 0$$

- if x is a non-degenerate critical point of *f*, then the number λ of negative eigenvalues of *H* is called the ***index*** of x.
- The definition of critical points is *local* and *sensitive* to small perturbations of the surface.

Morse functions

- ✓ On any smooth compact manifold there exist Morse functions.
- ✓ On a compact manifold, any Morse function has only a finite number of critical points.
- ✓ Morse functions are everywhere dense in the space of all smooth functions on the manifold.
- ✓ The set *S* of all simple Morse functions is everywhere dense in the set of all Morse functions.

Morse Lemma & critical points

- ✓ **Morse Lemma.**
  In a neighbourhood of each non-degenerate critical point x, the function *f* can be expressed as:

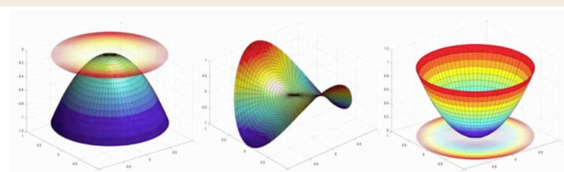
$$f = f(x) - y_1^2 - \ldots - y_\lambda^2 + y_{\lambda+1}^2 + \ldots + y_n^2$$

  where λ is the *index* of the critical point.

- ✓ **Euler formula.**

  #maxima - #saddles + #minima= χ(M).

Critical points classification



$f = -x^2 - y^2$	$f = -x^2 + y^2$	$f = x^2 + y^2$
Maximum λ=2	**Saddle λ=1**	**Saddle λ=0**

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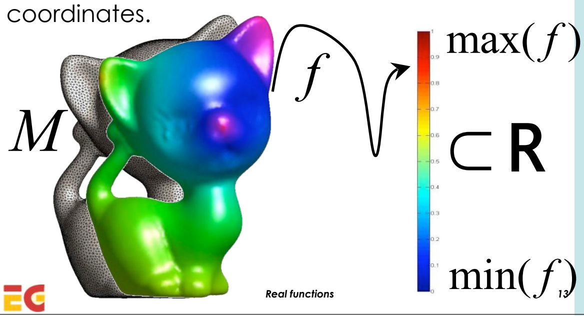
Functions on 3D shapes: discrete case

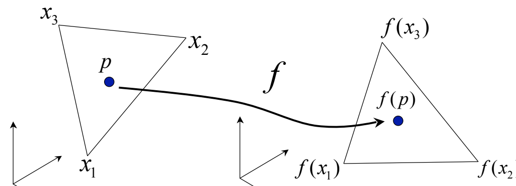
Define *f* on the mesh vertices and extend *f* to the edges and faces by using barycentric coordinates.



Real functions 13

Linear approximation: barycentric coordinates

f is uniquely determined by its values on the surface vertices of *M*.

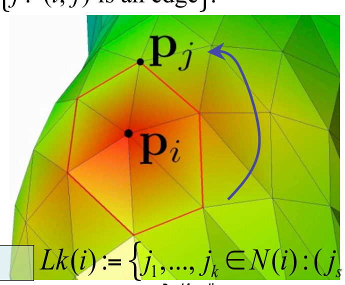


$$p = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \Leftrightarrow f(p) = \lambda_1 f(x_1) + \lambda_2 f(x_2) + \lambda_3 f(x_3)$$
$$\lambda_i \geq 0, i = 1,2,3, \lambda_1 + \lambda_2 + \lambda_3 = 1$$

Real functions 14

Critical points classification [Ban67]

- Each vertex p_i of *M* is classified according to the values of *f* on its 1-**star,** which is defined as $N(i) := \{ j : (i,j) \text{ is an edge} \}$.



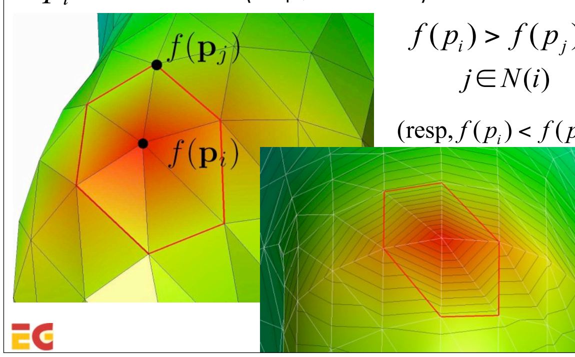
Link of i $Lk(i) := \{ j_1, \ldots, j_k \in N(i) : (j_s, j_{s+1}) \text{ edge} \}$

Real functions 15

Critical points: minimum/maximum

- p_i is a **maximum** (resp, minimum) if

$$f(p_i) > f(p_j)$$
$$j \in N(i)$$
$$(\text{resp}, f(p_i) < f(p_j))$$

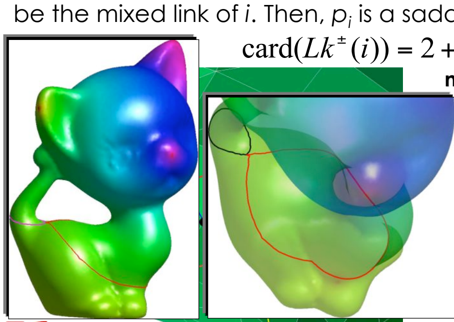


Critical points: saddle

- Let $Lk^{\pm}(i) := \{ (j_s, j_{s+1}) \in Lk(i) : f(j_{s+1}) > f(p_i) > f(j_s) \}$ be the mixed link of *i*. Then, p_i is a saddle if

$$\text{card}(Lk^{\pm}(i)) = 2 + 2m, \quad m \geq 1$$

multiplicity



17

Critical point properties: discrete case [Ban67,Ban70]

- If *f* is **general** (ie, *f(x) ≠ f(y)*, whenever x and y are distinct vertices of M), then the critical points of *(M,f)*
  - satisfy the Euler formula

$$\chi(M) = \#\text{minima-}\#\text{saddles+}\#\text{maxima},$$

  where saddles are counted with their multiplicity *m*
  - are located where the topological changes of (*M,f*) happen.

Real functions 18

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Examples

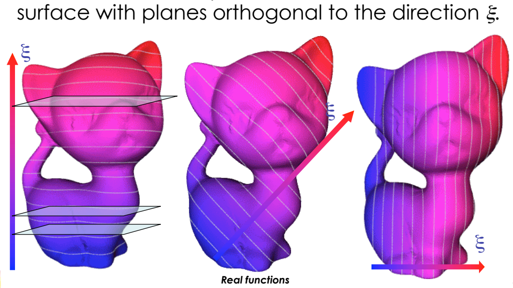
- Common choices of *f* are:
  - Height and elevation
  - Distance functions:
    - Euclidean-based
    - geodesic-based
  - Curvature-based functions
  - Local diameter
  - Laplacian-based functions:
    - Harmonic functions
    - Laplacian eigenfunctions
  - …

Evaluating the properties of *f*

- **Saliency:** ability to measure the shape features we are focusing on.
- **Smoothness:** behavior of *f* wrt the *nature* of its critical points.
- **Stability** wrt discretization and computation.
- **Robustness:** low variation of the *f* values wrt small changes of the shape.
- **DoF and heuristics:** number and type of parameters involved in the definition and/or computation of *f*.
- **Efficiency:** computational cost.
- **Invariance:** number/type/position of the critical points and the shape of the level-sets are "invariant" wrt a group of transformations.

Height function [Ban70,FK97]

- Given a direction ξ, the height function value at $x \in M$ with respect to ξ is defined as $f_\xi(x) := \langle x, \xi \rangle$.
- The level sets correspond to the intersection of the surface with planes orthogonal to the direction ξ.



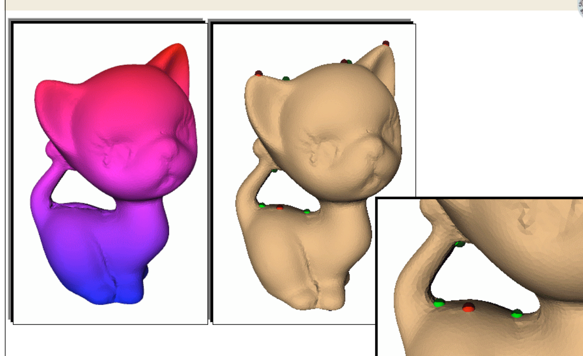
Height function

- **Saliency:** *f* is able to identify the shape features of *M* along the direction ξ.
- **Smoothness:**
  - Critical points are points whose normal is parallel to the direction ξ.
  - Almost all height functions are Morse (ie, the critical points are non degenerate).
- **Stability:** exact evaluation/computation of *f* and interpolation on the faces/edges of *M*.

Height function

- **Robustness:** the computation of *f* is robust, while its critical points aren't (see example).
- **DoF and heuristic:** the choice of ξ.
- **Efficiency:** computational cost for computing the function value
  - at one vertex: *O(1)*
  - on the whole *M*: *O(n)*, (*n*=#mesh vertices).
- **Invariance**: the function *f* is
  - "invariant" to translations
  - dependent on rotations: the recognized properties depend on the chosen direction.

Height function: robustness example



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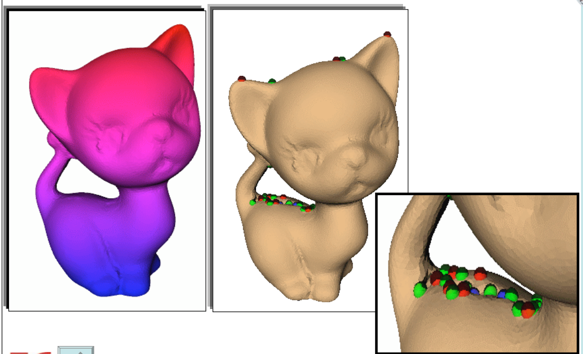
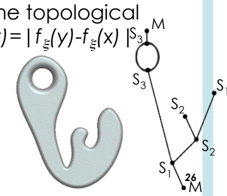
Height function: robustness example



Elevation [AEHW04]

- For any point x of M, there exists at least one direction ξ such that x is a critical point of the height functions f_{ξ} and $f_{-\xi}$.
- Then, for every $\xi \in S^2$
  - let x, y be two critical points of the height function wrt the direction ξ,
  - if x, y are paired according to the topological persistence, then $pers(x)=pers(y)=|f_{\xi}(y)-f_{\xi}(x)|$
  - the elevation is defined as

    Elevation(x)=pers(x).

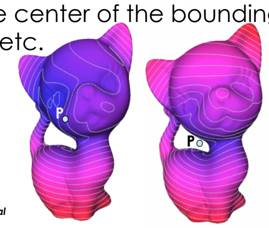


Elevation

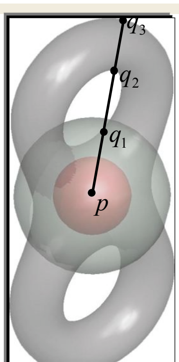
- **Saliency:** *f* identifies the depression and protrusions of *M* wrt any normal direction.
- **Smoothness:** *f* is continuous and smooth almost everywhere.
- **Efficiency:** the overall computational cost for
  - finding the persistence pairs: $O(n\log^2 n)$
  - classifying critical points: $O(n^5\log^2 n)$.
- **Invariance:** *f* is invariant to translations and rotations.

Euclidean distance from a point [FK97]

- $f(x) := \| x-p \|_2$
- The level sets correspond to the intersection of the surface with a set of spheres centered at the point *p*.
- Common choices of the point are the barycenter of *M*, the center of the bounding sphere or box of *M*, etc.

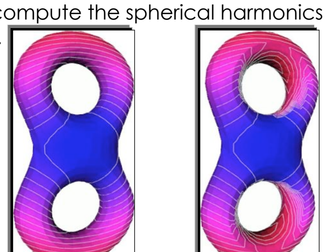


Radial distance from a point [SV01]



$$f(q) := \max\{r \geq 0 : r(q-p) \in M\}$$

In an analogous way, *f* can be defined on the unit sphere and used to compute the spherical harmonics of *f*.



Euclidean distance from *p* | Radial distance from *p*

Euclidean distance from a point

- **Saliency:** maxima and minima are located on protrusions and concavities wrt *p*.
- **Smoothness:** almost all distance functions from a point are Morse.
- **Stability:** exact computation at the mesh vertices.
- **Robustness:** the computation of *f* is robust, while its critical points aren't (see example).
  - For instance, the distance from the barycenter: due to its dependence on all the vertices, the barycenter is not affected by small perturbations of *M*.

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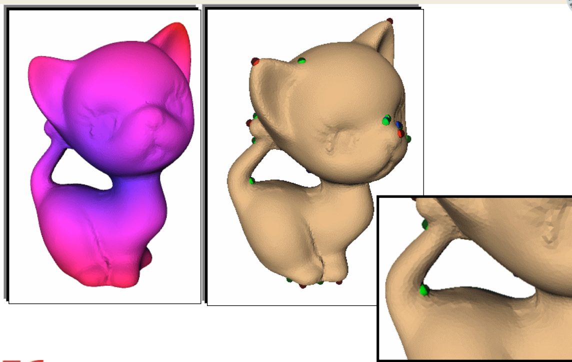
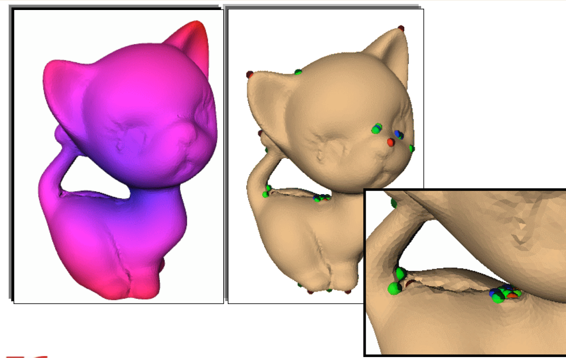
Euclidean distance from a point: robustness example



Real functions 31

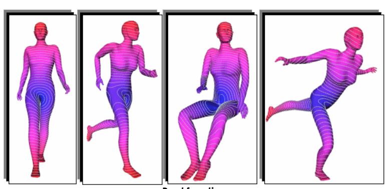
Euclidean distance from a point: robustness example



Real functions 32

Euclidean distance from a point

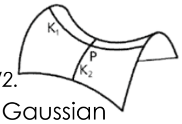
- **DoF and heuristics:** the point *p*.
- **Efficiency:** *f* is computed in *O(n)* time.
- **Invariance:**
  - *f* is "invariant" to translations and rotations
  - *f* is suitable to distinguish among different poses.



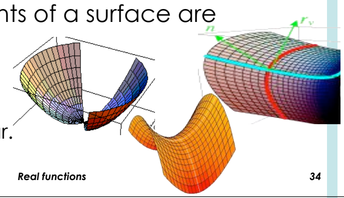
Real functions 33

Curvature-based function [GCO06,MPS*04, ZP01]

- The principal curvatures k_1 and k_2 at a point $p \in M$ measure the maximum and minimum bending of a surface at *p:*
  - the Gaussian curvature $K=k_1 k_2$
  - the Mean curvature $H=(k_1+k_2)/2$.



- According to the sign of the Gaussian curvature, the points of a surface are classified as
  - elliptic
  - hyperbolic
  - parabolic or planar.



Real functions 34

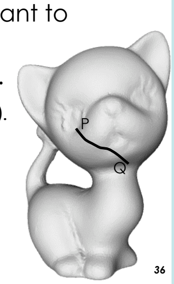
Curvature-based function

- **Saliency:** is provided by the characterization of the local shape as elliptic, hyperbolic, parabolic/planar.
- **Smoothness:** related to the differentiability degree of *M*.
- **Stability:** a coarse surface sampling and an irregular connectivity badly affect the discretization of the curvature.
- **Robustness:** low degree.
- **DoF and heuristics:** the size of the neighborhood used to compute *K* and *H*.
- **Efficiency:** depends on the size of the neighborhood; at least O(*n*) wrt the 1-star.
- **Invariance:**
  - K is intrinsic, ie it is invariant wrt isometries
  - H is extrinsic and depends on the surface embedding.

Real functions 35

Geodesic distance: definition and properties

- Given two points $p,q \in M$, the geodesic distance *g(p,q)* is the length of the shortest path between *p* and *q*.
- The geodesic distance is invariant to isometric transformations.
- The shortest path is not unique.
- Exact computation in $O(n^2 \log n)$.
- Approximations:
  - Dijkstra [VL99]: $O(n \log n)$
  - [SSK*05]: $O(n \log n)$
  - Fast marching [KS98]: $O(n)$.



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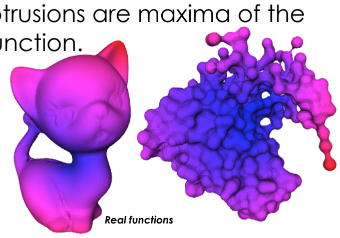
Average geodesic distance [HSKK01]

- The mapping function is defined as

$$f(p) = \int_{v \in M} g(p,v)dm,$$

where g represents the geodesic distance
- Surface protrusions are maxima of the mapping function.



Real functions 37

Average geodesic function

- Discretized using a set of base points $\{b_1, \ldots, b_n\}$ instead of all mesh vertices:

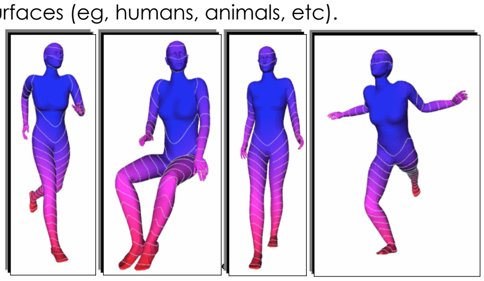
$$f(p) = \sum_i g(p,b_i)area(b_i),$$

where $area(b_i)$ is the influence region of b_i.
- It has been extended to consider also the angle variation along a path [KT03].

Real functions 38

Average geodesic distance

- **Saliency:** f discriminates protrusions of M.
- **Invariance:** f is invariant to isometries, that is, it does not distinguish among different poses of articulated surfaces (eg, humans, animals, etc).



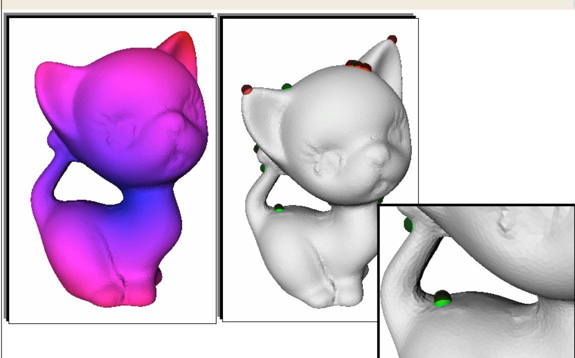
39

Average geodesic distance

- **Smoothness:** f is smooth.
- **Stability:**
  - the discretization and computation depend on the chosen algorithm, eg., Dijkstra [VL99], [SSK*05], fast marching [KS98]
  - generally, a coarse surface sampling and an irregular connectivity affect the discretization of the geodesic distance
  - the instabilities are averaged by the integral in the definition of f.
- **Robustness:** f is robust to local shape changes (see example).
- **DoF and heuristics:** choice of the base points used to discretize the integral.
- **Efficiency:** depends on the discretization and number of base points. It is computationally expensive using the Dijkstra's algorithm with all vertices as base points: $O(n^2 logn)$.

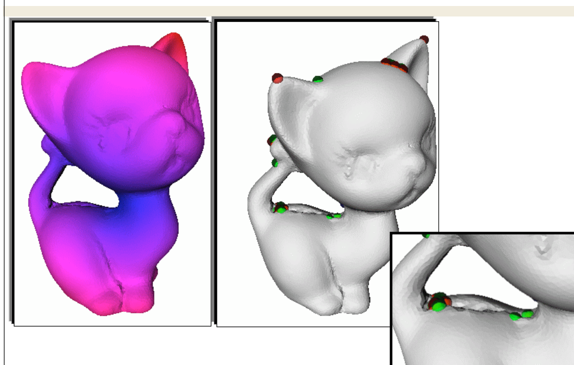
Real functions 40

Average geodesic distance: robustness example



Real functions 41

Average geodesic distance: robustness example



Real functions 42

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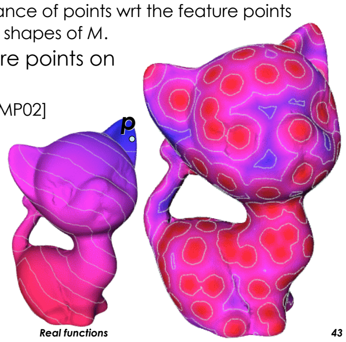
Geodesic distance from feature points [MP02,VL99]

- The geodesic distance can be used to
  - measure the importance of points wrt the feature points
  - characterize tubular shapes of *M*.
- Choice of the feature points on the surface:
  - curvature extrema [MP02]
  - user-defined [VL99]
  - uniform sampling.



Topological distance from curvature extrema [MP02]

- Let *p* be the centroid of a high-curvature region, we define

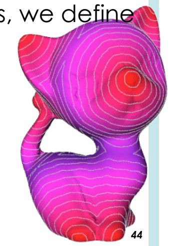
  $g_p(q)$:=min{*k*: $q \in k$-neighborhood}.

- Given $\{p_1,\ldots, p_k\}$ *k* feature points, we define g as:

$$g(q) := \min\left\{g_{p_1}(q), \ldots, g_{p_k}(q)\right\}.$$

and

$$f(q) := g_{\max} - g(q).$$



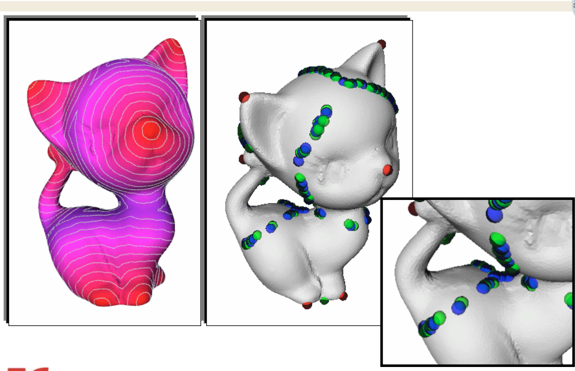
Topological & geodesic distance from curvature extrema

- **Saliency:** *f* discriminates protrusions, especially those that include the curvature extrema as feature points.
- **Smoothness:** low degree.
- **Stability:**
  - topological distance: since *f* is discretized using the connectivity of *M*, the neighborhood expansion is computationally stable
  - geodesic distance: the stability of *f* is affected by the mesh connectivity.

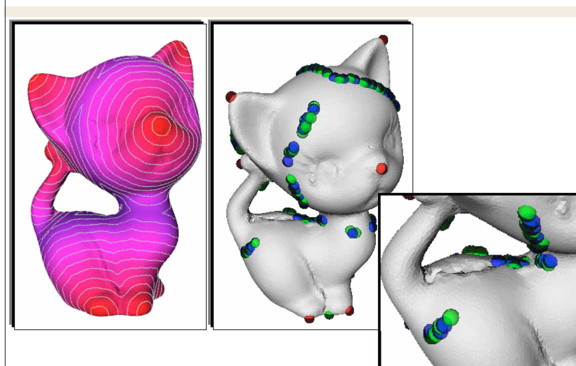
Topological & geodesic distance from curvature extrema

- **Robustness:** the geodesic (resp, topological) distance from feature points is robust wrt small geometric and connectivity (resp, geometric) changes.
- **DoF and heuristics:** choice of the feature points.
- **Efficiency:** the computational cost of the topological expansion is O(*n*) and O(*nlogn*) for the geodesic distance.
- **Invariance:**
  - topological distance: *f* is invariant wrt any transformation that preserves the mesh connectivity
  - geodesic distance: *f* is invariant to isometric transformations.

Topological distance from curvature extrema: robustness example



Topological distance from curvature extrema: robustness example



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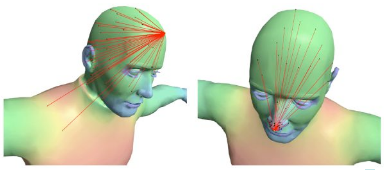
Local diameter shape function [GSC07]

- On a smooth surface, the *exact diameter* of a shape at a point *p* is the distance to the antipodal point of *p* wrt the direction opposite to the normal at *p*.
- The *local diameter* function at *p*
  - is a statistical measure of the diameters in a cone around the direction opposite to the normal at *p*.
  - requires close



Local diameter shape function

- **Saliency:** morphological characterization of the shape in terms of relative size of its parts.
- **Smoothness:** no guarantees of smoothness for the local shape diameter: it may fail at sites of branching or in particular visibility cones.
- **Stability:** yes.
- **Robustness:** robust to deformations that do not locally alter the shape volume.
- **DoF and heuristics:** no DoF; heuristics drive the statistical sampling of the diameters.
- **Efficiency:** $O(n^2)$.
- **Invariance:**
  - invariant to translations and rotations
  - Invariance to pose changes is forced by averaging the values of *f* at the vertices of *M* wrt the values of neighbors.

Harmonic functions [PP93,TAU00,DBG*06]

Smooth functions with a (generally) low number of critical points are achieved by solving the Laplace equation with Dirichlet boundary conditions.

$$\begin{cases} \Delta f(p_i) = 0, i \in I & \text{Laplace equation} \\ f(p_i) = \alpha_i, i \in I^C & \text{Dirichlet boundary conditions} \end{cases}$$

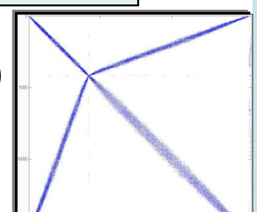
$$I \subseteq \{1,\dots,n\} \qquad \Delta := div \circ \nabla$$

interior vertices

Laplacian matrix of a triangle mesh

$$\Delta f(p_i) := \sum_{j \in N(i)} [f(p_j) - f(p_i)] w_{ij}$$

$$L_{ij} = \begin{cases} \sum_{j \in N(i)} w_{ij} & i = j \\ -w_{ij} & (i,j) \text{ edge} \\ 0 & \text{else} \end{cases} \qquad Lf = 0$$

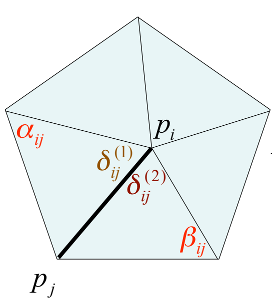


$L \in \mathbb{R}^{n \times n}, rank(L) = n-1$

$$\bar{L}\,\bar{f} = b$$

Sub-matrix of *L* | Unknown values of *f* | Know right-hand vector

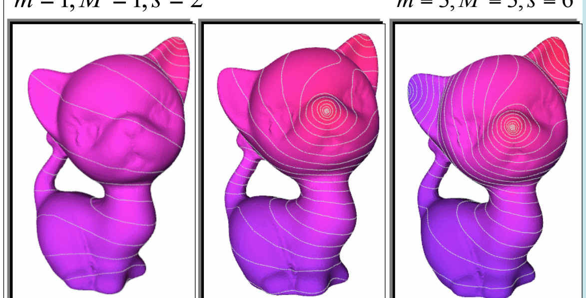
Discretization: weights [Flo97,PP93,...]



$$w_{ij} := \begin{cases} \dfrac{\cot \alpha_{ij} + \cot \beta_{ij}}{2} \\ \dfrac{\tan(\delta_{ij}^{(1)}/2) + \tan(\delta_{ij}^{(2)}/2)}{\| p_j - p_i \|_2} \\ \dfrac{1}{d_i} \\ 1 \end{cases}$$

Harmonic functions: locality

$m = 1, M = 1, s = 2$ $\qquad$ $m = 3, M = 3, s = 6$



$m = 2, M = 2, s = 4$

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Harmonic functions

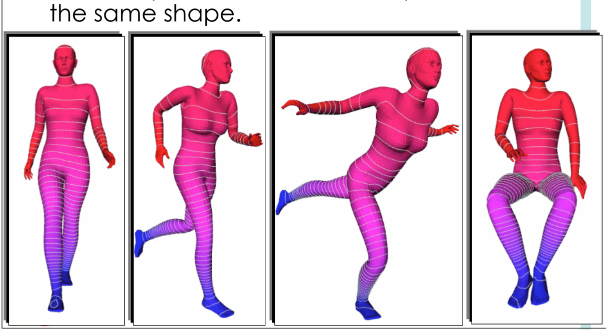
- **Saliency:**
  - the choice of the maxima and minima of *f* (→Dirichlet conditions) on feature regions guarantees their characterization through (*M*,*f*)
  - topological saliency: if f has only *1* min and *1* max, then the saddle points are located on the topological handles of *M*.
- **Smoothness**:
  - the number of critical points depends on the Dirichlet boundary conditions and the genus of the input surface
  - using as Dirichlet boundary conditions *1* max & *1* min guarantees to build a harmonic function *f* with a minimal number of critical points (ie, 2*g* saddles)
  - $f \in_{\approx} C^2(M, \mathbf{R})$.

Harmonic functions

- **Stability:** the Laplace operator is local and uses only the *1*-star of each vertex. Numerical instabilities might be introduced by its discretization:
  - the cotangent weights might be negative: $\alpha_{ij} + \beta_{ij} > \pi$
  - the mean-value weights are always positive and more stable than the cotangent weights.
- **Robustness:** the computation and the properties of *f* are robust wrt changes of the surface and connectivity that do not make unstable the discretization of the Laplace operator (see example).

Harmonic functions: robustness

- Harmonic functions with the same Dirichlet boundary conditions: different postures of the same shape.



Harmonic functions

- **DoF and heuristics:** the choice of the Dirichlet boundary conditions.
- **Efficiency:**
  - solution of a sparse linear system *O(nlogn)*
  - changing the Dirichlet boundary conditions does not require to re-build the Laplacian matrix.
- **Invariance:**
  - *f* is invariant wrt isometries
  - with constant weights, *f* is affine invariant.

Eigensystem of the Laplacian matrix [NGH04,RWP06]

- The spectrum of the Laplacian matrix associated to *M* enables to define a set of functions "intrinsically" defined by the input shape.
- Since *L* is symmetric, it has a real eigensystem

$$Lx_i = \lambda_i x_i, \quad i = 1, \ldots, n$$

and

$$\forall y \in \mathbf{R}^n, \; y = \sum_{i=1}^{n} \alpha_i x_i.$$

Spectrum of the Laplacian Matrix

- Eigenvalues:

$$0 = \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$$

- Eigenvectors:

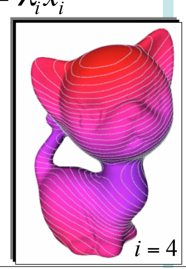
$$(x_i, \lambda_i), \quad Lx_i = \lambda_i x_i$$

- i-th function

$$f_i : M \to \mathbf{R}$$

$$f_i(p_k) = \sqrt{\lambda_i}\,[x_i]_k$$

$$i = 2, \ldots, n$$



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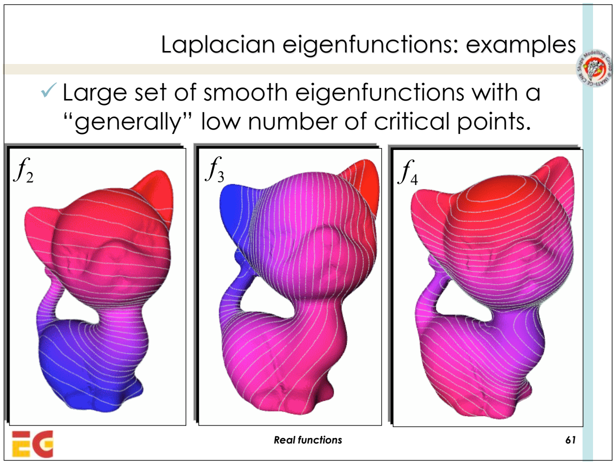

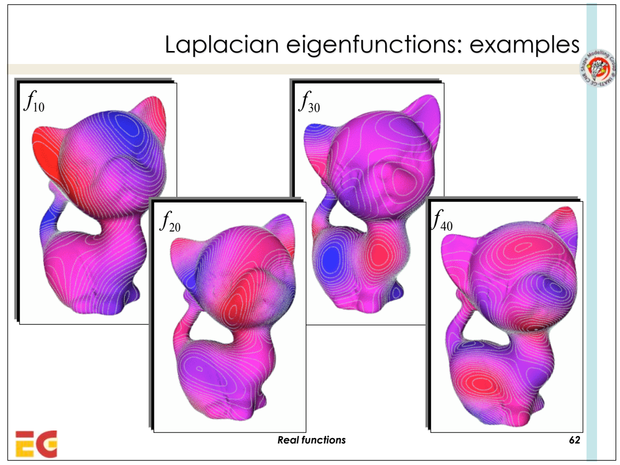

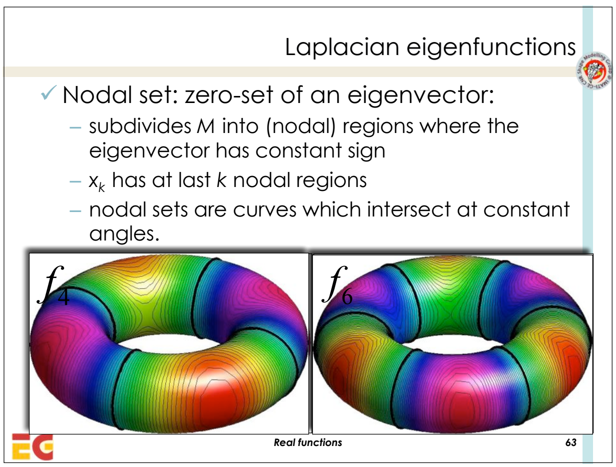

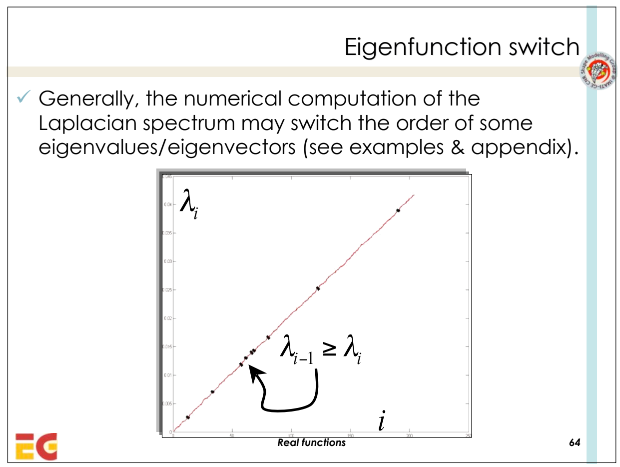

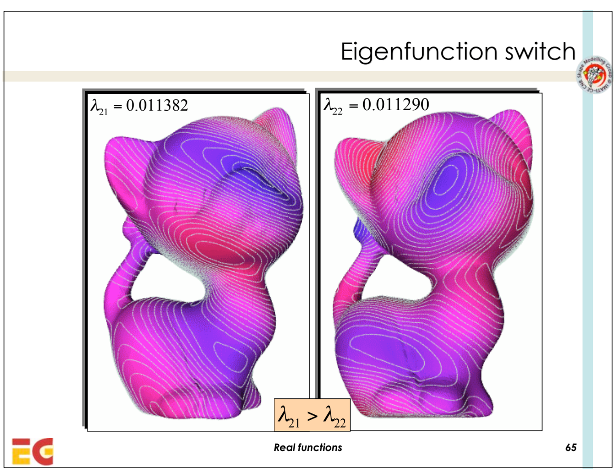

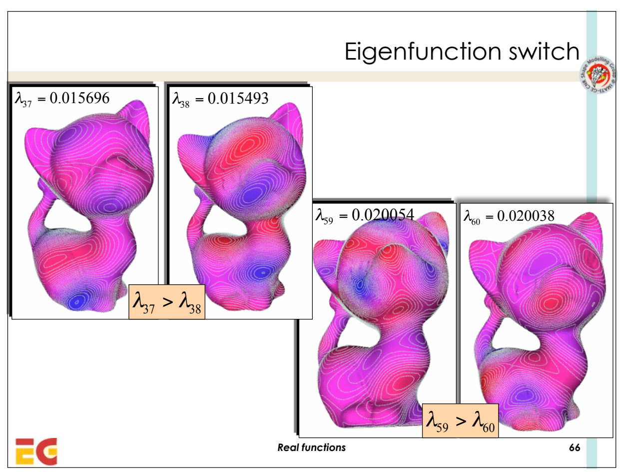

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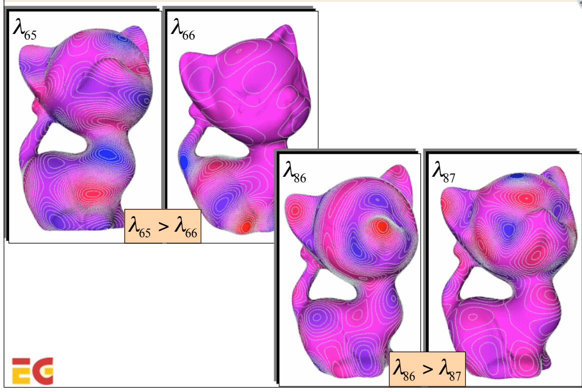
Eigenfunction switch



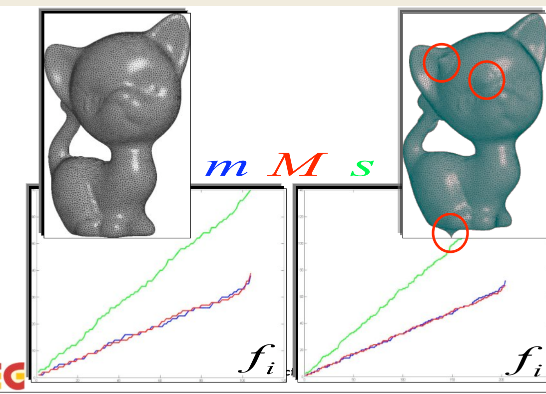
Laplacian- eigenfunctions

- **Saliency:** each function is intrinsically defined by *M*.
- **Smoothness:** the first eigenvectors correspond to smooth and slowly varying functions, while the last ones show rapid oscillations.
- **Stability:**
  - the discretization of the Laplace operator is local and uses only the *1*-star of each vertex
  - numerical instabilities might be introduced by its discretization
  - the switch of the eigenfunctions might happen regardless the mesh discretization.

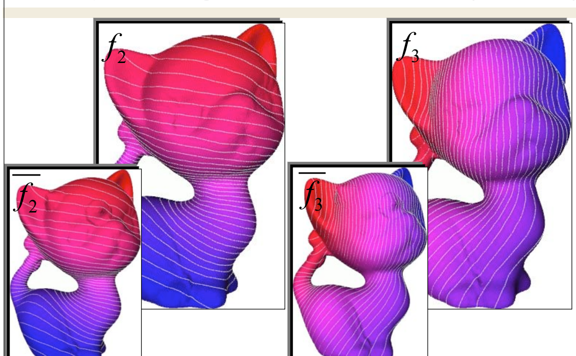
Laplacian eigenfunctions

- **Robustness:** the computation and the properties of *f* are robust wrt changes of the surface and connectivity that do not make unstable the discretization of the Laplace operator (see examples).
- **DoF and heuristics:**
  - choice of f_i among (*n-1*) non-trivial functions
  - sign of the eigenvectors.
- **Efficiency:** O(*nlogn*), O(n^2) depending on the sparsity of *L*.
- **Invariance:**
  - *f* is invariant wrt isometries
  - with constant weights, *f* is affine invariant.

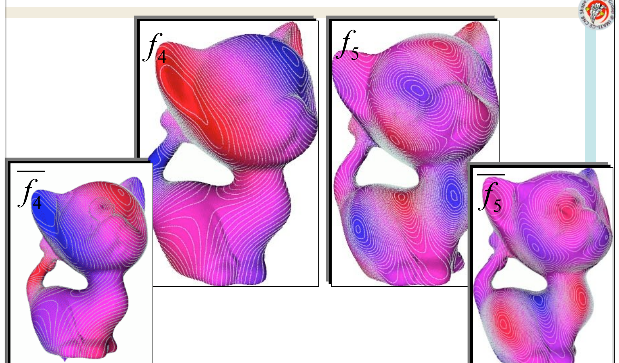
Laplacian eigenfunctions: robustness (critical points)



Laplacian eigenfunctions: robustness (level sets)



Laplacian eigenfunctions: robustness (level sets)



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Questions?

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Appendix: Perturbation Theory for Eigenvalues and Eigenvectors [GV89]

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Perturbation theory: general case

- Right eigenvector $Ax = \lambda x$
- Left eigenvector $y^* A = \lambda y^*$
- If A is diagonalizable, $y_i^* x_j = 0, i \neq j$
- Consider the matrix

$$A_\varepsilon := A + \varepsilon B, \qquad |b_{ij}| \leq 1$$

with right eigenvector $x_i(\varepsilon)$ and eigenvalue $\lambda_i(\varepsilon)$.
Pb: Which relation exists between the eigensystem of A, A_ε ?

Eigenvalue perturbation: general case

- For each eigenvalue, the following relation holds

$$\left|\frac{\lambda_i(\varepsilon) - \lambda_i}{\varepsilon}\right| \rightarrow \frac{\|B\|_2}{s(\lambda_i)}, \qquad s(\lambda_i) := |y_i^* x_i|.$$

- Then, the above estimation is:
  - proportional to the l_2-conditioning number of the perturbation matrix *B*
  - inversely proportional to the angle between the left and right eigenvectors.

Eigenvalue perturbation: general case

- We note that

$$s(\lambda_i) = |\underbrace{y_i^* x_i}_{\geq 0}| \leq \|y_i\|_2 \|x_i\|_2 = 1.$$

- The term $s(\lambda_i)^{-1}$ is called **conditioning number** of the eigenvalue λ_i.
- Then, an eigenvalue is **well-conditioned** iff its conditioning number is not close to zero.

Eigenvalue perturbation: Laplacian matrix

- If the input surface is closed (or with boundary + virtual edges), the Laplacian matrix is symmetric and

$$y_i \equiv x_i, \quad s(\lambda_i) = 1, \quad i = 1, \dots, n.$$

- Each eigenvalue is well-conditioned and

$$\left|\frac{\lambda_i(\varepsilon) - \lambda_i}{\varepsilon}\right| \rightarrow \|B\|_2, \quad \varepsilon \rightarrow 0.$$

- The variation of the eigenvalues depends only on the l_2-norm of the perturbation matrix *B*.

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Eigenvector perturbation: general case

- For the *i*-th eigenvector, we have

$$\left\| x_i(\varepsilon) - x_i \right\|_2 \leq \varepsilon \sum_{j \neq i}^{n} \left| \frac{y_j^* B x_i}{(\lambda_i - \lambda_j) s(\lambda_j)} \right| + O(\varepsilon^2).$$

- Then, the bound depends on:
  - the conditioning number of **each** eigenvalue $s(\lambda_i)$
  - the differences $\lambda_i - \lambda_j$
  - the factors $\beta_{ij} := y_j^* B x_i$.

Real functions 79

Eigenvector perturbation: general case

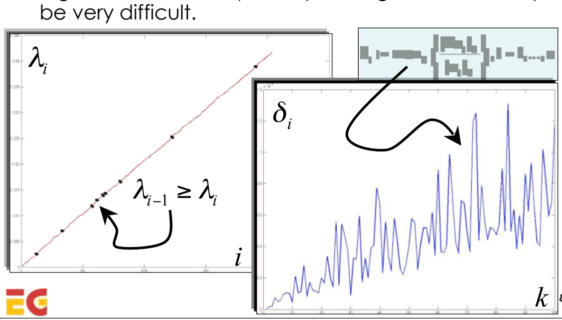
- The perturbation in the eigenvector is proportional to the conditioning number of the whole set of eigenvalues.
- If the eigenvalues are close to one another, we may have difficulties in computing the eigenvectors.
- Let *A* have distinct eigenvalues. If for some eigenvalue $s(\lambda) < 1$, then there exists a matrix *E* such that λ is a repeated eigenvalue of (*A*+*E*) and

$$\frac{\|E\|_2}{\|A\|_2} \leq \frac{s(\lambda)}{\sqrt{1 - s(\lambda)^2}}.$$

Real functions 80

The perversity theorem does not hold

- Then, even if the eigenvalues are distinct, if one eigenvalue is ill-conditioned, the computation of the eigenvalues, and especially the eigenvectors, may be very difficult.



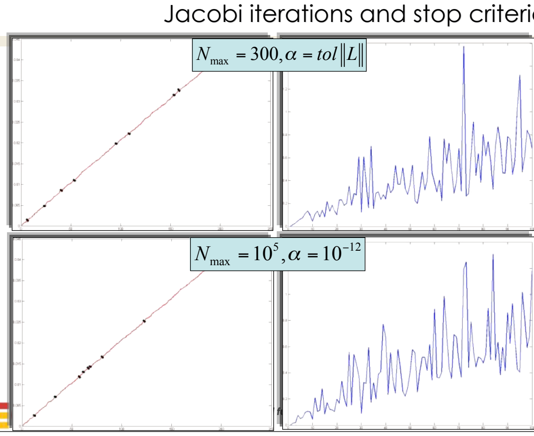
81

Jacobi iterations and stop criteria

- The (first or last) elements of the eigensystem of the input matrix are evaluated by using the Jacobi method with 2 stop criteria:
  - max. number of iteration $N_{\max}$
  - approximation threshold α.
- Increasing $N_{\max}$ and reducing α do not avoid the switching of eigenvalues and eigenvectors.

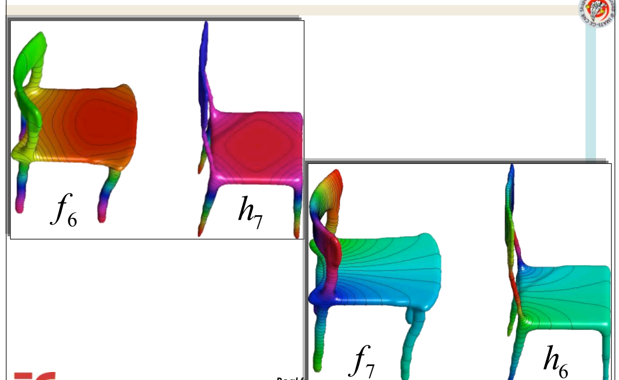
Real functions 82

Jacobi iterations and stop criteria



83

Eigenfunction "switch" on different shapes



Real f

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Eigenfunction switch: discussion

- The switch of the eigenfunctions
  - can happen among the eigenfunctions of the same surface;
  - is strictly correlated to the computation of the eigensystem;
  - a "good" geometry and connectivity (wrt the computation of the entries of L) do not guarantee to avoid the switch of the eigenfunctions.

Real functions 85

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