3D Shape Description and Matching Based on Properties of Real Functions

Shape Descriptors

Eurographics 2007 Tutorial T12

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Outline

- Descriptors parametric with respect to *f*:
  - Reeb graph
  - Size theory tools
  - Persistent homology tools
  - Descriptors based on spherical decompositions
    - Spherical harmonics
    - Spherical wavelets
- Descriptors linked to a specific *f*:
  - Bending invariant surface signature
  - Spectral embedding
  - Pose-oblivious shape signature
  - Shape-DNA

Properties to be discussed

- **Saliency**: ability to capture the essential features of the shape
- **Conciseness**: ability to minimize the memory needed to store the description while maximizing the amount of information
- **Robustness** wrt small changes of the shape
- **Uniqueness** and **completeness**
- **Invariance** to transformation groups
- **DoF and heuristics** used in the construction of the descriptor
- **Input:** hypotesis and restrictions
- **Efficiency**: computational cost

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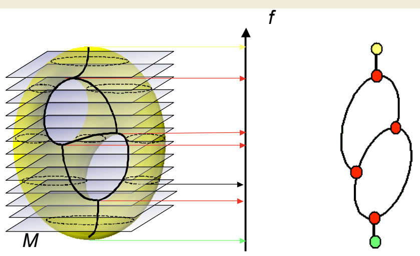
Reeb graph [Ree46]



Reeb graphs are used to store the evolution of the level sets of the mapping function ***f***

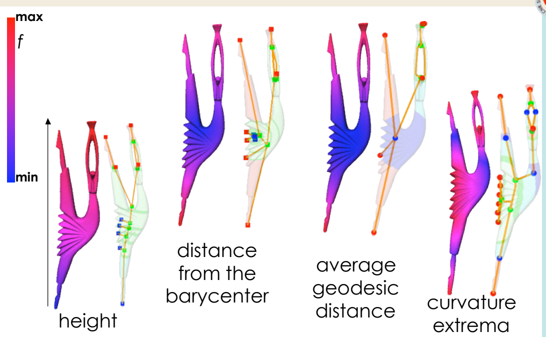
Reeb graph [Ree46]

Let M be a compact *n*-manifold and *f*: M→R a simple Morse function. Let "~" be the **equivalence relation**:

(P, f(P)) ~ (Q, f(Q)) ⇔ *f(P) = f(Q)* and P and Q are in the same connected component of $f^{-1}(f(P))$

The quotient space on MxR is a finite and connected **simplicial complex K of dimension 1**, such that the counter-image of each vertex of dim 0 of K is a singular connected component of the level sets of f, and the counter-image of the interior of each simplex of dim 1 is homeomorphic to the topological product of one connected component of the level sets by R

RGs when the function *f* varies

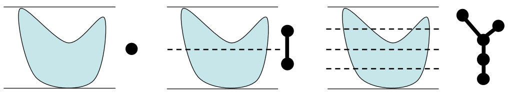


Reeb graph based representations

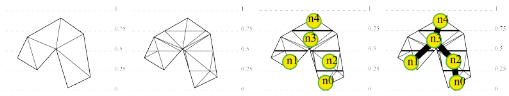
- Different proposals for descriptors induced by the Reeb graph:
  - Multiresolution Reeb graph (MRG) [HSKK01, BSR06]
  - Augmented Multiresolution Reeb graph (aMRG) [TS05]
  - Extended Reeb graph (ERG) [BFS00,Bia04,BMSF06]

Multiresolution Reeb graph [HSKK01]

- Provides a hierarchical graph encoding



- The graph is extracted inserting contours in a progressive manner



- The area *A* of a region and the relative size *L* of the interval of *f* are associated as attributes to nodes

Augmented Multiresolution Reeb graph [TS05]

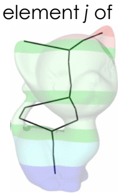
- The descriptor is the same but the nodes are enriched with attributes storing other geometric measures, that are related to the spatial extent of the regions associated to the nodes:
  - relative volume
  - statistic measure of the chords
  - Koenderink shape index
  - statistic orientation of the triangle normals
  - statistic on the texture (when available)

Multiresolution Reeb graphs [HSKK01,TS05]

- **Saliency**:
  - MRGs capture geometrical and topological properties of the shape; if the average geodesic distance is used, MRGs capture protrusions of the shape
  - the graph attributes influence the saliency
- **Conciseness**: very good conciseness properties due to the synthesis of the information (geometry and topology) in an attributed graph
- **Robustness**: no theoretical results
- **DoF and heuristics**: the resolution has to be chosen
- **Uniqueness**: fixed the resolution, the MRG is unique
- **Completeness**: no
- **Invariance**: inherit the invariance properties of *f*
- **Input**: manifold, closed triangle meshes
- **Efficiency**: the cost of the graph extraction is O(*n*+*k*), where *k* is the number of vertices added during the construction

Extended Reeb graph [Bia04]

- Founds on an extended Reeb equivalence
  - let $f : M \rightarrow \mathbf{R}$ be a real valued function;
  - let $\mathbf{I} = \{(f_{min}, f_1), (f_h, f_{max}), (f_i, f_{i+1}), i=1\ldots h-1\} \cup \{f_{min}, f_1, \ldots f_h, f_{max}\}$ be a partition of $[f_{min}, f_{max}]$;
  - the extended Reeb equivalence between $P, Q \in M$ is given by:
    - $f(P)$, $f(Q)$ belong to the same element j of $\mathbf{I}$;
    - P, Q belong to the same connected component of $f^{-1}(j)$.



Extended Reeb graph [BMSF06]

- Each arc can be oriented using the growing direction of the mapping function: the ERG is a direct acyclic graph
- Each ERG node *v* can be labelled with attributes measuring properties of regions or subparts associated with *v* (eg, using spherical harmonics)
- Store for each ERG arc *e* the number of slices traversed by the arc (arc length)

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Extended Reeb graph [Bia04]

- **Saliency**:
  - captures geometrical and topological properties of the shape
  - the geometric embedding influences the saliency
  - it preserves the topology of the manifold
- **Conciseness**: very good conciseness properties due to the synthesis of the information (geometry and topology) in an attributed graph
- **Robustness**: no theoretical results
- **DoF and heuristics**: the partion has to be chosen
- **Uniqueness:** fixed the partition, the ERG is unique
- **Completeness:** no

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Extended Reeb graph [Bia04]

- **Invariance**: inherites the invariance properties of *f*
- **Input:** orientable 2-manifold represented by triangle meshes
- **Efficiency**: the cost of the graph extraction is *O(n+k)* (where *k* is the number of vertices added during the construction)

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Size theory and size functions [Fro90]

- Two shapes (topological spaces *and* real functions) are similar if there exist a homeomorphism between the spaces that almost preserves the function values
- We measure how well such properties are preserved via the natural pseudo-distance, that measures the infimum of the variation of the functions values moving from a space to the other through homeomorphisms
- Two shapes are similar if their the natural pseudo-distance is small
- We get information about the natural pseudo-distance from size functions, a mathematical tool providing a lower bound for the natural pseudo-distance

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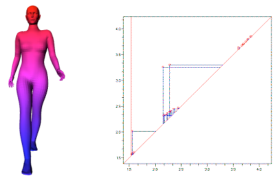
Size theory and size functions [Fro90, FM99,FL99]

- The (1-dimensional) size function of the size pair (M,φ), with $\varphi : M \to \Re$, is the function

$$\ell_{(M,\varphi)} : \{(x,y) \in \Re^2 : x < y\} \to N$$

that takes each *(x,y)* to the number of components of the lower level set M_y, that contain at least a point of M_x



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Multidimensional size functions [BCF*07]

$(\mathcal{M}, \vec{\varphi})$ size pair, with $\vec{\varphi} : \mathcal{M} \to \mathbb{R}^k$;

for every $\vec{x} = (x_1, \ldots, x_k)$, $\vec{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k$

- $\vec{x} \preceq \vec{y}$ $(\vec{x} \prec \vec{y}) \overset{def}{\iff} x_i \leq y_i$ $(x_i < y_i)$, $i = 1, \ldots, k$;
- $\mathcal{M}_{\vec{x}} = \{P \in \mathcal{M} : \vec{\varphi}(P) \preceq \vec{x}\}$;

$\Delta^+ = \{(\vec{x}, \vec{y}) \in \mathbb{R}^k \times \mathbb{R}^k : \vec{x} \prec \vec{y}\}$;

Definition

The (multidimensional) size function of the size pair $(\mathcal{M}, \vec{\varphi})$ is the function $\ell_{(\mathcal{M},\vec{\varphi})} : \Delta^+ \to \mathbb{N}$ that takes each $(\vec{x}, \vec{y})$ to the number of connected components of $\mathcal{M}_{\vec{y}}$ that contain at least a point of $\mathcal{M}_{\vec{x}}$.

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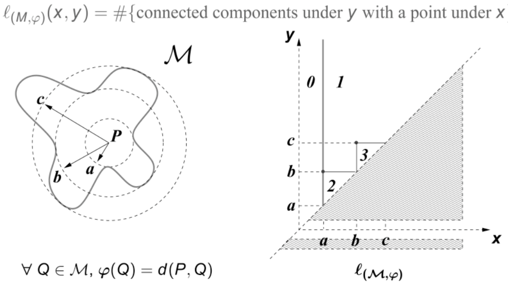
Example with k=1

$\ell_{(M,\varphi)}(x, y) = \#\{\text{connected components under } y \text{ with a point under } x\}$



$\forall\, Q \in \mathcal{M}$, $\varphi(Q) = d(P, Q)$ $\qquad \ell_{(\mathcal{M},\varphi)}$

(Symbols $\vec{x}, \vec{y}, \vec{\varphi}$ are replaced by x, y, φ).

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Example with k=1

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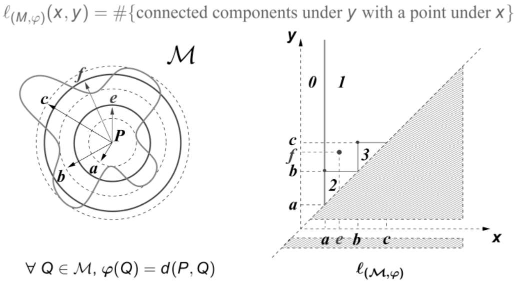

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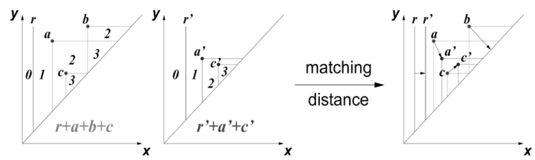
K=1: Representation and matching [FL01,dAFL06]

Each 1-dimensional size function can be represented by a formal series of points representing vertices of triangular region in Δ^+.



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K=1: Stability of matching distance [dAFL06]

✓ Matching Stability Theorem:

$$\max_{P \in \mathcal{M}} |\varphi(P) - \psi(P)| \leq \epsilon \Rightarrow d_{match}(\ell_{(\mathcal{M},\varphi)}, \ell_{(\mathcal{M},\psi)}) \leq \epsilon.$$

Small changes in the mapping functions imply small changes in the size functions: robustness wrt perturbation of the data

✓ Lower bound for the natural pseudo-distance: Let λ be the value of the matching distance between the two size functions $\ell_{(M,\varphi)}$ e $\ell_{(N,\psi)}$. Then

$$d((M,\varphi),(N,\psi)) \geq \lambda.$$

This guarantees a link between the comparison of size functions and the comparison of shapes

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Multidimensional size functions [BCF*07]

✓ PROBLEMS WITH k>1:
- How to extend the representation as formal series of points and lines?
- How to compare size functions with k>1? A direct approach involves working in a domain of $\Re^{2k}$
- How to obtain stability in computation?

✓ SOLUTION: there exists a foliation in half-planes of the domain of multidimensional size functions s.t. on each leaf of the foliation the multidimensional size function coincides with a particular 1-dimensional size function; this allow to difine a stable multidimensional matching distance

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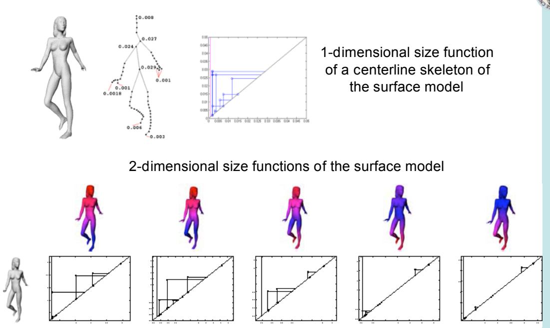
Size functions [BGSF06,BCF*07]



Size functions [Fro90, BCF*07]

- **Saliency**:
  - captures geometrical and topological properties of the shape
  - connection with the comparison of topological spaces in terms of natural pseudo-distance
- **Conciseness**: very concise combinatorial description
- **Robustness**: theoretically proven robustness wrt small shape changes
- **Unique** but not **complete**

Size functions [Fro90, BCF*07]

- **Invariance**: inherits invariance properties from the underlying measuring functions
- **DoF** and **heuristics**: none
- **Input**: representation of shapes as size graphs (from discrete or discretized objects, eg., meshes and grids)
- **Efficiency**: the computational complexity is $O(n\log n + m\alpha(2m+n,n))$, with n, m the number of nodes and arcs of the size graph, and α the inverse of the Ackermann function

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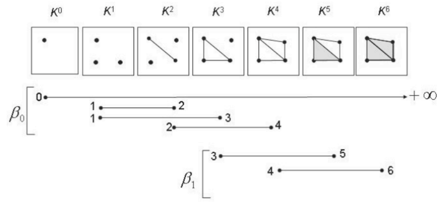
Persistent homology [ELZ02]

- The idea of persistent homology is to control the placement of topological events in a growing space and assess their relevance according to their life-time
- Persistence represents the life-time of cycles in the growing filtration; this life-time is represented by persistence intervals



Persistent homology [ELZ02]

- Given a growing complex K, represented by a filtration

$$\{K^i\}_{i=,\dots,n},\ K^n = K,\ K^i \text{subcomplex of } K^{i+1}$$

the j-persistent k-th homology group of K^i is a group isomorphic to the image of the homomorphism

$$\eta_{ij}^{\ k} : H_k(K^i) \rightarrow H_k(K^{i+j})$$

induced by the inclusion of K^i into K^{i+j}

- The persistent homology of a growing complex can be represented by a set of intervals, called **persistence intervals**: a P – interval is a pair

$$(i,j),\quad i,j \in \mathbb{Z} \cup +\infty,\ 0 \le i < j$$

such that there exists a cycle that is completed at the level i of the filtration and remains non-bounding until the level **j**

Persistent homology and barcodes [CGCZ05]

- The shape of a complex *K* can be described by filtering the complex by the increasing values of a real function
- Idea: construct a new complex strictly related to *K*, namely the tangent complex *T(K)* (closure of the space of all tangents to all points in K), and filter it with the function computing the curvature at a point along a tangent direction
- The barcode of the shape is the set of *P* – intervals for the filtered tangent complex

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Spherical harmonics [VSR01]

- Idea: build multi-resolution feature vectors using the Fourier expansion of a function defined on the sphere
- Represent the spherical function *f*: $S^2 \rightarrow \mathbf{R}$ (eg. the spherical extent function, measuring the extent of the object in given directions) as

$$f(\theta,\varphi) = \sum_{l\geq 0} \sum_{|m|\leq l} a_{l,m} Y_l^m(\theta,\varphi)$$

- Feature vectors can be extracted from the first rows of coefficients, thereby providing a multiresolution approach

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Spherical harmonics [KFR03]

- Represent a function *f* defined on the sphere through its spherical harmonics and consider the vector of energies (i.e. frequency norms)

$$SH(f) = \{\|f_0(\theta,\varphi)\|, \|f_1(\theta,\varphi)\|, \ldots,\}$$

with f_l the frequency components

$$f_l(\theta,\varphi) = \sum_{m=-l}^{l} a_{lm} Y_l^m(\theta,\varphi)$$

- Let R be a rotation; then it holds:

$$SH(R(f)) = SH(f)$$

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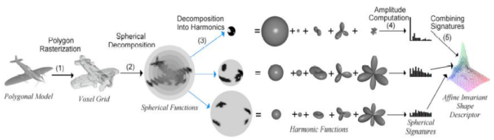
Spherical harmonics [KFR03]

- Extension to voxel description:
  - Restrict the voxel grid to a collection of concentric spheres
  - Represent each spherical restriction in terms of the energy of its frequency decomposition, thus obtaining a 1D descriptor
  - The final descriptor resulting from the analysis of spheres with different radii is a 2D grid indexed by radius and frequency



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Spherical wavelets [LTN06]

- The problem of the sensitivity of the sampling of the spherical function to latitude-longitude parametrization of the sphere is addressed
- A rotation invariant sampling is proposed, relying on the flat octahedron parametrization of the sphere
- A Spherical Wavelet Transform is applied to the spherical shape function
- Resulting descriptors:
  - Matrix of wavelet coefficients (SWC)
  - L1 energy-based feature vector (SWEL1)
  - L2 energy-based feature vector (SWEL2)

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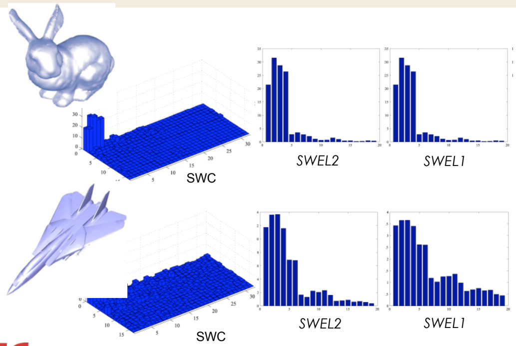
Spherical wavelets [LTN06]



Spherical harmonics and wavelets [VSR01,KTR03,LTN06]

- **Saliency**: captures the geometrical properties expressed by the spherical function
- **Conciseness**: very concise descriptors (feature vectors or matrices)
- **Robustness**: robustness wrt small changes of the spherical function derived from the decomposition properties
- **Unique**, but not **complete**, since a finite number of coefficients or energies is taken into account
- **Invariance**:
  - Wrt translation
  - Wrt rotation: [VSR01] requires alignment, [KTR03] is invariant only to rotations applied to the sampled input, in [LTN06] SWC requires alignment, while SWEL1 and SWEL2 are rotation invariant

Spherical harmonics and wavelets [VSR01,KTR03,LTN06]

- **DoF** and **heuristics**:
  - Sampling and voxelization
  - number of frequency components
  - [LTN06] requires the choice of the wavelet transform
- **Input**: meshes (also poligon soups) and grids
- **Efficiency**: the computational complexity in [KTR03] is O(n^3) with n the size of the voxel grid

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Bending invariant surface signature [EK03]

- Geodesic distances between surface points are invariant to surface bending
  - Idea: use geodesic distances to define an isometrical embedding of a surface in a small dimensional Euclidean space, in which geodesic distances are approximated by Euclidean ones
  - Method: apply a MultiDimensional Scaling (MDS) procedure on a matrix of geodesic distances between surface points

Bending invariant surface signature [EK03]

- Sample with *n* vertices a given triangulated surface, via iterative Voronoi sampling, and build an *n* x *n* matrix *D*

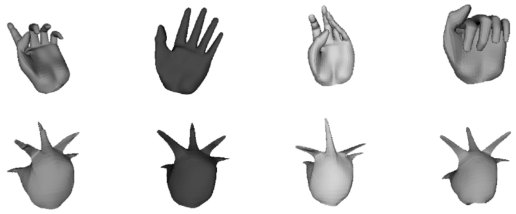
$$D_{ij} = (\delta_{ij})^2$$

with δ_{ij} the geodesic distance between vertices *i*,*j* computed following the fast marching on triangulated domains algorithm

- Define a dimension *m* for the Euclidean embedding space and apply MDS on the matrix *D*, yelding an $\Re^m$ *n* x *m* matrix whose rows define the coordinates in of the points of the signature surface

Bending invariant surface signature [EK03]

- These two steps define a bending invariant descriptor, that allows to translate the problem of matching non-rigid objects in various posture into a simpler problem of matching rigid objects



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Bending invariant surface signature [EK03]

- **Saliency**:
  - enhances protrusions captured by the geodesic distance
  - scalable amount of captured shape information, related to the dimension of the embedding
- **Invariance** wrt isometry
- Not **unique,** due to the randomly chosen starting point in the sampling stage, and not **complete**

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Bending invariant surface signature [EK03]

- **Input**: triangulated surfaces
- **DoF and heuristics**:
  - choice of the dimension of the sampling and the embedding
  - choice of the specific MDS algorithm (classical, least squares, fast)
- **Efficiency**:
  - Computing the matrix requires O(n^2), with n the number of sampled vertices
  - the MDS algorithm is at most O(nN), with N number of iterations

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Spectral embedding [JZ07]

- Ideas similar to [EK03] are developed, introducing a descriptor suitable to compare articulated objects
- The matrix D is a matrix involving a Gaussian of width σ

$$D_{i,j} = e^{-\frac{\delta_{ij}^2}{\sigma}}$$

with geodesic distances approximated through an heuristic
- The embedding in $\Re^m$ is given by the first m eigenvectors of the matrix, computed via Nyström approximation

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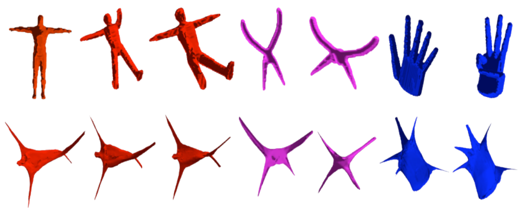
Spectral embedding [JZ07]

- The descriptor is given by the embedded surface or by the matrix first eigenvalues



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Spectral embedding [JZ07]

- **Saliency**:
  - enhances protrusions captured by the geodesic distance
  - the possibility to use affinity matrices based on different functions (e.g. Euclidean or combined distances) is suggested
  - Scalable amount of shape information, related to the embedding dimension
- **Robustness**: sensitiveness to outliers in the data; problems related to eigenmode switching and eigenmode sign assignment have to be faced
- **Invariance** wrt isometries

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Spectral embedding [JZ07]

- **Unique** but not **complete**
- **Input**: triangulated surfaces (possibly to be repaired)
- **DoF and heuristics**:
  - sampling rate
  - embedding dimension
  - Gaussian width
  - heuristic to compute the geodesic distance
- **Efficiency**: $O(Nnlogn+N^3)$ operations required to compute and eigen-decompose the matrix *D*, with *n* the number of vertices of the mesh and *N* the number of sampled points

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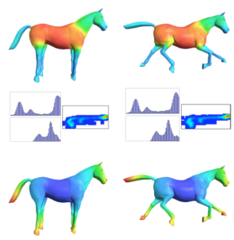
Pose-oblivious shape signature [GCO06]

- It combines in a 2D histogram the values of two scalar functions defined on the surface of the 3D shape:
  - local diameter function (DF)
  - centricity function [HSKK01]
- Each histogram bin with values (*x*,*y*) contains the approximated probability of a point on the surface to have a DF value of *x* and a CF value of *y*



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Pose-oblivious shape signature [GSCO06]

- **Saliency**:
  - use of pose-oblivious shape features
  - volume-function values (DF)and geometric positioning indicator (CF)
  - insensitiveness to topology
- **Conciseness**: very concise descriptor (64x32 histogram)
- Not **unique** and not **complete**
- **Robustness:** mainly related to the quantization in constructing the histogram and the properties of the functions
- **Invariance** properties derive from those of the functions
- **DoF and heuristics** are those involved in the DF computation (hard-coded parameters), plus the size of the histogram
- **Input:** (almost) watertight meshes
- **Efficiency**: depends on the computation of the functions

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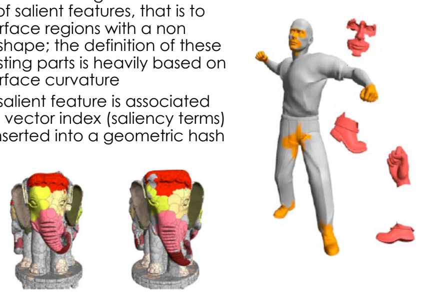
Salient geometric features [GCO06]

- Partial matching of surface meshes is addressed through the definition a set of salient features, that is to say surface regions with a non trivial shape; the definition of these interesting parts is heavily based on the surface curvature
- Each salient feature is associated with a vector index (saliency terms) and inserted into a geometric hash table



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- ✓ Descriptors linked to a specific *f*:
  - Bending invariant surface signature
  - Spectral embedding
  - Pose-oblivious shape signature
  - Shape-DNA

Shape Descriptors 66

Shape DNA [RWP06]

- ✓ The shape DNA is the beginning of the spectrum of the Laplace – Beltrami operator, defined for real valued functions on Riemannian manifolds:

Given a Riemannian *n*- manifold M and $f : M \to \Re$
the Laplace – Beltrami operator is

$$\Delta f := div(grad\, f)$$

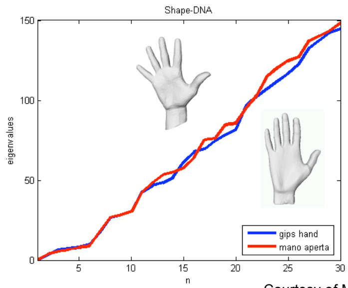
(different from the discrete Laplacian on graphs)

$$\text{Shape DNA} = \{\lambda_0 \le \lambda_1 \le \ldots \le \lambda_m\} \in \Re^m{}_{\ge 0}$$

with λ_i eigenvalues of the Helmholtz equation $\Delta f = -\lambda f$

Shape Descriptors 67

Shape DNA [RWP06]



Courtesy of Martin Reuter

Shape Descriptors 69

Shape DNA [RWP06]

- ✓ **Saliency**:
  - description of geometrical and topological properties intrinsic to the object
  - scalable amount of captured shape information, related to the dimension of the cropped spectrum
- ✓ **Robustness**: continuously dependent on shape deformations
- ✓ **Unique** but not **complete**, since there exist isospectral but not isometrical shapes
- ✓ **Invariance** wrt isometry
- ✓ **DoF and heuristics**: choice of the number of eigenvalues in the cropped spectrum
- ✓ **Input**: parametric surfaces, polygonal meshes, solid polyhedra; independent w.r.t. parametrization
- ✓ **Efficiency**: the eigenvalue computation is the most time consuming step

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