Mathematical Tools for 3D Shape Analysis and Description

SGP 2013 Graduate School

tools and concepts, part II
Andrea Cerri and Silvia Biasotti

- mathematical concepts
  - basics on algebraic topology
  - simplicial complexes
  - homology
  - Morse theory
- concepts in action
  - comparing shapes
  - persistent topology
  - Reeb graphs (by Silvia)

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isometries, but not only...

- topological spaces and functions to model shapes and their properties as pairs \((X, f)\);
- suitable for dealing with stability/robustness;
- are there alternatives to isometries to solve other sets of problems?

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algebraic topology & homology

- Associates algebraic invariants with topological spaces;
- Classifies topological spaces up to homeomorphisms;
- Homology is an option.

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approach: to decompose a topological space into simple pieces easier to study;

q-simplices are the building blocks:
- combinatorial aspect: relations among simplices;
- geometric aspect: related to their embedding in the Euclidean space.

a simplicial complex is the combinatorial structure, generated by q-simplices;

simplices and faces

a q-simplex in \( \mathbb{R}^n \), with \( n \geq q \), is the convex hull of \( q + 1 \) affinely independent points.

a face of a q-simplex is the convex hull of a subset of the set of its points.

dimension of a simplicial complex

the dimension of \( K \) is the highest among the dimensions of its simplices:
- triangle meshes are 2-complexes;
- tetrahedral meshes are 3-complexes.
a *q*-chain is a sum of all *q*-simplices in *K*,
\[ \sigma = \sum_i \lambda_i \sigma_i, \lambda_i \in \mathbb{Z}_2 = \{0,1\} \]
- a curve on a mesh is a 1-chain
- a surface patch is a 2-chain

the *q*-chain space \( C_q(K) \) is the vector space generated by all the *q*-simplices in *K*;

**boundary operator (more formally)**

\[ \partial_q : C_q(K) \to C_{q-1}(K) \]
proves the boundary of a boundary is null

**cycles and boundaries**

- is a *q*-cycle if its boundary is 0;
- is a *q*-boundary if is a boundary of a \((q+1)\)-chain;

**Intuition:**

Homology formalizes this idea
an element of $H_q(K)$ is an equivalence class containing homologous $q$-cycles;

the rank of $H_q(K)$ is the $q$-th Betti number of $K$;

For 3D data:
- $\beta_0 = \#$ components;
- $\beta_1 = \#$ tunnels;
- $\beta_2 = \#$ voids;

the homology $H_*(K)$ is a topological invariant.

Let $S$ be a connected, orientable surface without boundary. The genus $g$ of $S$ is
- the maximum number of cuttings along non-intersecting closed simple curves which can be cut along the surface without disconnecting it...
- ... or half of the first Betti number $\beta_1(S)$.

The genus is a topological invariant.

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functions and critical points (I)

$f: M \to \mathbb{R}$ smooth function on a smooth manifold $M$ of dimension $n$:
- $p \in M$ is critical if $df_p$ is the zero map, i.e. $\frac{df}{dx_1}(p) = \cdots = \frac{df}{dx_n}(p) = 0$;

A critical point $p \in M$ is non-degenerate if the Hessian $H_f(p)$ is non-singular, i.e.
$$\det \left( H_f(p) \right) = \det \left[ \frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right] \neq 0$$

If $p$ is a non-degenerate critical point of $f$, the index $\lambda_p$ of $p$ is defined as:
$$\lambda_p = \#(\text{negative eigenvalues of } H_f(p))$$

$f: M \to \mathbb{R}$ is a Morse function if all of its critical points are non-degenerate;
Morse functions & critical points: Two results

- For \( c \in \mathbb{R} \), set \( M_c = \{ p \in M : f(p) \leq c \} = f^{-1}((-\infty, c]) \).
- How/when does the topology of \( M_c \) changes?
- To combine the topology of \( M \) with the quantitative measurement provided by \( f \):
  - \( f \) is the lens to look at the properties of \((M, f)\).
  - Different choices of \( f \) provide different invariants.
- To construct a general framework for shape characterization which is parameterized wrt the pair \((X, f)\).

Morse theory can be used for:

- Comparing objects' shapes can be done with respect to some "relevant properties";
- To model a shape we consider pairs \((X, f)\) s.t.
  - \( X \) represents the object;
  - \( f : X \rightarrow \mathbb{R} \) is a function and describing the relevant properties of the object.

content

- Mathematical concepts
  - Basics on algebraic topology
  - Simplicial complexes
  - Homology
  - Morse theory

- Concepts in action
  - Comparing shapes
  - Persistent topology
  - Reeb graphs (by Silvia)

references

- W. Massey, Algebraic topology: An Introduction, Brace&World Inc., 1967
- J. Milnor, Morse theory, Princeton University Press, New Jersey, 1963
- C. Kosniowski, A First Course in Algebraic Topology, Cambridge University Press, 1966
How can we compare two pairs \((X, f), (Y, g)\)?

Idea: Use a metric able to quantify and measure the deformation of \(X\) into \(Y\), in terms of \(f\) and \(g\).

\[
d((X, f), (Y, g)) = ?
\]

formally: **Natural pseudo-distance** \(d\)

\[
d((X, f), (Y, g)) = \inf_{\text{homomorphisms } h} \max_{x \in X} \|f(x) - g(h(x))\|_{\infty},
\]

\(h\) varying among all homomorphisms from \(X\) to \(Y\).

Problem: the natural pseudo-distance \(d\) is difficult to compute; we need tools to get information about it;

Persistent topology allows us get lower bounds for \(d\) by means of suitable shape descriptors;

Instead of comparing pairs \((\text{space}, \text{functions})\), we can compare the associated descriptors.

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A filtration is a nested sequence \(X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n = X\), e.g. the sub-level sets of a function \(f: X \to \mathbb{R}\).

Topological exploration along a filtration of \(X\), looking for important topological events.

Measure the lifespan of homology classes along the filtration:

\[
f_x
\]

Encode the birth level \(i\) and the death level \(j\) of a homology class by a point \((i, j)\).
Changing functions produces different information.

Stability with respect to the bottleneck distance*: $d_B(\text{Dgm}(f), \text{Dgm}(g)) \leq \inf_{\alpha \in \mathcal{X}} \max_{x \in \mathcal{X}} \|f(x) - g(h(x))\|_\infty$

This implies resistance to noise; it is a lower bound for the natural pseudo-distance.

* Cohen-Steiner et al. (2005), Chazal et al. (2009), d’Amico et al. (2010)...

Retrieval of trademark images [C., Ferri, Giorgi 2006]

Approach:
- for each image, battery of 25 descriptors;
- each distance between descriptors of the same type induces a metric on the database;
- combine all 25 metrics (max, sum…) to obtain a final similarity score.
Other possible applications...

- Point cloud data analysis [Chazal et al. '09]
- 3D segmentation [Skraba et al. '10]

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Other possible applications...

- data simplification [Bauer, Lange, Wardetzky '12]
- 3D (textured) shape analysis and comparison [Biasotti et al. '13]

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Reeb graph

- Reeb graphs store the evolution of the level sets of the mapping function $f$

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Reeb graph definition

- let $M$ be a compact $n$-manifold, $f: M \rightarrow \mathbb{R}$ a simple Morse function, and given "~" the equivalence relation
  $(P, f(P)) \sim (Q, f(Q)) \iff f(P) = f(Q)$ and $P$ and $Q$ are in the same connected component of $f^{-1}(f(P))$
- the quotient space on $M \times \mathbb{R}$ is a finite, connected simplicial complex $K$ of dimension 1, such that
  - the counter-image of each vertex of $K$ is a singular connected component of the level sets of $f$
  - the counter-image of the interior of each simplex of dim 1 is homemorphic to the topological product of one connected component of the level sets by $\mathbb{R}$

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multidimensional setting ($f: X \rightarrow \mathbb{R}^k$):
- multidimensional persistent homology group [Carlsson, Zomorodian '07];
- Multidimensional size functions [Biasotti et al. '08];
- Persistence spaces [C., Landi '13];
- ...

an historical perspective

- Size functions [Frosini '91];
- Persistent homology groups [Edelsbrunner, Letscher, Zomorodian '02];
- Vines and vineyards [Cohen-Steiner, Edelsbrunner, Morozov '06];
- Interval persistence [Dey, Wegner '07];
- Zig-Zag persistence [Carlsson, de Silva, Morozov '09];
- Persistent cohomology [de Silva, Morozov, Vejdemo-Johansson];
- ...

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overview of RGs when the function $f$ varies

- $f$ values
- height
- barycenter
- bounding sphere
- center
- integral
godesic
- curvature
- extrema
draw the Reeb graph with respect to the height function \( f \) of the following shapes

- RG properties
  - it provides a 1D structure of the shape
  - it describes the shape of an object under the lens of the function \( f \)
  - nodes and arcs depend on the number of critical points of \( f \)
  - it may be computed in \( O(n \log n) \) operations

- Reeb graph based representations
  - Reeb graph variations
    - contour trees (simply-connected domains)
    - component trees (gray-level images)
    - centerline skeletons (geodesic distance from a point)
  - for shape matching
    - Multiresolution Reeb graph (MRG), Hilaga et al. 2001
    - augmented Multiresolution Reeb graph (aMRG), Tung & Schmitt 2005
    - Extended Reeb graph (ERG), Biasotti et al. 2000

- extension to volume data
  - nodes of the ERG correspond to regions (either surfacic or volumic)
  - arcs code the adjacency among the parts
  - each arc can be oriented using the growing direction of the mapping function: the RG is a direct acyclic graph
  - with each ERG node, store spatial attributes measuring properties of the part associated to it
    - node embedding (Cartesian coordinates, x, y, z)
    - average value of \( f \) in the part
    - volume
    - area
    - some radii: min, max, average

- geometric embedding
geometric embedding

- each arc can be oriented using the increasing direction of \( f \): the RG is a direct acyclic graph
- nodes and arcs correspond to surface parts
- node and arc attributes are stored in terms of the spherical harmonic indices of the parts

part correspondence

- models with similar appearance

part correspondence

- objects with dissimilar global appearance

shape retrieval

- performance on the SHREC’07 watertight database

shape approximation and compression

- given the Reeb graph \( \text{RG}(M,f) \) of \( f \): Morse and simple, a rough shape chartification is obtained

1-strip contains one max or min
one boundary component

3-strips contains one saddle
three boundary components

\( f \) may assume different values on each of these three boundary components

138K

2.6MB

v=3.8K

138K

v=35K

t =69K
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wrap up
Silvia Biasotti

we have seen:
- how to represent a shape (topological spaces, manifolds, metric spaces, simplicial complexes...)
- how to analyze, describe, compare shapes (geodesics, curvature, diffusion geometry, Morse theory, algebraic machinery...)
- examples of different applications

we have investigated:
- space, invariance, property...
- theoretical assumptions vs real world assumptions

but if we go back to «real» shapes and problems, are we sure that everything works ok?
- services which could use the theories discussed
- discretization issues and applicability

complex shapes, complex problems

online repositories of 3D models
- Google 3D Warehouse
  http://sketchup.google.com/3dwarehouse/
- 3Dvia
  http://www.3dvia.com/search/
complex shapes, complex problems

✓ online repositories of 3D models
  – Google 3D Warehouse http://sketchup.google.com/3dwarehouse/
  – 3Dvia http://www.3dvia.com/search/
  – Turbosquid http://www.turbosquid.com/
  • “…search our stock catalog to get the 3D model you want, or use our Custom 3D modeling service for made-to-order 3D models. Join the world’s top artists who use TurboSquid 3D models in advertising, architecture, broadcast, games, training, film, the web, and just for fun”

complex shapes, complex problems

✓ 3D Object Retrieval & Content-based search
  – Princeton Shape Benchmark
  – Digital Shape Workbench v5.0 http://visionair.ge.imati.cnr.it/


to sum up: theory says ...

✓ 3D shapes maybe very complex and services we can imagine for sharing, searching, and even design new shapes are many, so...
✓ … how to get out of this maze?
✓ basics
  – the choice of one or another shape description should be guided by
    ▪ type of shapes and their variability/complexity
    ▪ invariants or properties to be preserved / captured
    ▪ topological spaces and functions are promising to model and reason about similarity in a mathematically well-formulated manner

pay attention to...

✓ … the right space
  ▪ rigid transformations (rotations, translations)
  ▪ Euclidean distances
  ▪ isometries
  ▪ Riemannian metric
  ▪ curvature (but unstable to local noise/perturbations)
  ▪ geodesics, diffusion geometry, Laplacian operators, etc
  ▪ local invariance to shape parameterizations
  ▪ conformal geometry
  ▪ similarities (i.e. scale operations)
  ▪ normalized Euclidean distances
  ▪ affinity (and homeomorphisms)
  ▪ persistent topology
  ▪ Morse theory
  ▪ size theory

pay attention to...

✓ … a suitable shape description
  – coarse coding (but fast)
    ▪ histograms
    ▪ matrices
    ▪ articulated shapes
    ▪ medial axes
    ▪ Reeb graphs
    ▪ overall global appearance
    ▪ silhouettes
    ▪ if shape loops are relevant
      ▪ persistent topology
      ▪ graph-based descriptions

pay attention to...

✓ importance of benchmarking and evaluation!
  – not only publishing papers but also to demonstrate real innovation and applicability potential
✓ Shape segmentation benchmark
  – SHape Retrieval Context http://www.aimatshape.net/event/SHREC
  – an annual event to to evaluate the effectiveness of 3D shape analysis algorithms
  – a multi-track event spanning
    ▪ different models: from watertight objects to triangle soups, from abstract shapes to medical data
    ▪ different tasks: from 3D retrieval to correspondence finding and segmentation


to sum up: benchmarking
to evaluate the stability of the algorithms under variations of abstract shapes characterized by
- smooth/sharp features,
- partial/global symmetries
- different genus

to create a new benchmark where to measure the capability of the retrieval methods to be invariant to
- noise addition and resampling
- deformations

9 transformations, 3 degrees each, for a total of 27 perturbations:
1. addition of Gaussian noise
2. different regular sampling
3. uneven sampling
4. & 5 two stretching
6. & 7 two non-uniform dilations
8. & 9 two non-uniform erosions

504 watertight models without self-intersections

recall-precision curves (whole dataset)

recall-precision curves (deformations)
conclusions

✓ the right tool for the right problem
  – mathematically sound methods
  – benchmarking & evaluation
✓ geometry, structure, similarity, context
  – is it possible to understand something about functionality or affordance?
  – machine learning vs geometric-reasoning
  – 3D query modalities
  – what if shape is influenced/modified by the context?

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at the end...

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	hank you for your attention!

these course notes are available at: http://www.ge.imati.cnr.it/training