Mathematical Tools for 3D Shape Analysis and Description

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Outline
- motivation
- mathematics and shape analysis challenges (11:35–11:45)
  - shape properties and invariants
  - similarity between shapes
- tools and concepts, part I (11:45–12:15)
  - topological spaces, functions, manifolds
  - metric spaces, isometries, curvature, geodesics
  - Gromov-Hausdorff distance
  - concepts in action
- tools and concepts, part II (14:00–15:00)
  - basics on topology, homology and Morse theory
  - natural pseudo-distance
  - concepts in action
- conclusions (15:00–15:15)

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3D Shape Analysis and Description

Where are we now?
- technology today
  - plenty of 3D acquisition techniques
  - hardware for visualizing 3D on the desktop
  - computer networks: fast connections, low cost
  - 3D printers: not only mock-ups but even end products

rendering, acquiring, transmitting, “materializing” 3D content is now feasible in specialized as well as unspecialized contexts

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3D media
- professionals
  - Product Modeling & Design
  - Cultural Heritage
  - Gaming
  - Spatial Data
  - Simulation
  - Medicine
  - Bioinformatics
  - Architecture
  - Archaeology
- non professionals
  - 3D social networking
  - fabbing
  - ...
shape and geometry

✓ “... all the geometrical information that remains when location, scale, and rotational effects are filtered out from an object” [Kendall 1977]

shape and similarity

✓ “...the form of something by which it can be seen (or felt) different by something else” [Longman Dictionary of Contemporary English]

shape, similarity & the observer

✓ things possess a shape for the observer, in whose mind the association between the perception and the existing conceptual models takes place [Koenderink 1990]

understanding, reasoning, similarity is a cognitive process, depending on the observer and the context

shape and view points

objects and similarity

geometric congruence

structural equivalence

functional equivalence

semantic equivalence
objects and similarities

- geometric congruence
- structural equivalence
- functional equivalence
- semantic equivalence

shape and description

- shape descriptions reduce the complexity of the representation; their choice depends on
  - type of shapes and their variability/complexity
  - invariants or properties

shape descriptions

- different shapes should have different descriptions
  - different enough to discriminate among shapes
- a shape may not be entirely reconstructed from its description

what's invariance?

- invariance = the descriptor does not change for a given object under a class of transformations
  - a property P is invariant to a transformation T applied to an object O iff
    \[ P(T(O)) = P(O) \]

shape descriptions and similarity

- similarity in what sense?
  - defining appropriate similarity measures between shapes

mathematics: shape description and similarity

- similar shapes with respect to what?
  - shape descriptions, to code the aspects of shapes to be taken into account and manage the complexity of the problem

- similarity in what sense?
  - transformations among the shapes that we consider irrelevant to the assessment of the similarity
  - invariants or properties

shape and description

- shape descriptions reduce the complexity of the representation; their choice depends on
  - type of shapes and their variability/complexity
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what's invariance?

- invariance = the descriptor does not change for a given object under a class of transformations
  - a property P is invariant to a transformation T applied to an object O iff
    \[ P(T(O)) = P(O) \]
things are not that easy...

✓ to deal with the complexity at a hand...
✓ we need tools to reason about
  – connectivity, interior, exterior and boundary
  – measuring shape properties and invariants
  – well-posedness
  – robustness and stability
  – distance and proximity
  – etc.…

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tools and concepts, part I

content

✓ tools and concepts
  – topological spaces
  – continuous and smooth functions
  – homeo- and diffeomorphisms
  – manifolds
  – transformations
  – metric spaces
  – intrinsic properties
    • curvature
    • conformal structure
    • geodesic distances
    • Laplace-Beltrami operator
  – Gromov-Hausdorff distance
✓ concepts in action

why topological spaces?

✓ to represent the set of observations made by the observer (e.g., neighbor, boundary, interior, projection, contour):
✓ to reason about stability and robustness

topological spaces

✓ a topological space is a set $X$ together with a collection $T$ of subsets of $X$, called open sets, satisfying the following axioms:
  1. $X, \emptyset \in T$
  2. any union of open sets is open
  3. any finite intersection of open sets is open

✓ the collection $T$ is called a topology on $X$

why functions?

✓ to characterize shapes
✓ to measure shape properties
✓ to model what the observer is looking at
✓ to reason about stability
✓ to define relationships (e.g., distances)
continuous and smooth functions

- let $X, Y$ topological spaces, $f: X \to Y$ is continuous if for every open set $V \subseteq Y$ the inverse image $f^{-1}(V)$ is an open subset of $X$.
- let $X$ be an arbitrary subset of $\mathbb{R}^n$: $f: X \to \mathbb{R}^m$ is called smooth if $\forall x \in X$ there is an open set $U \subseteq \mathbb{R}^n$ and a function $F: U \to \mathbb{R}^m$ such that $F = f|_U$ on $X \cap U$ and $F$ has continuous partial derivatives of all orders.

why manifolds?

- to formalize shape properties
- to ease the analysis of the shape
  - measuring properties walking on the shape
  - look at the shape locally as if we were in our traditional euclidean space
  - to exploit additional geometric structures which can be associated to the shape

manifold

- manifold without boundary
  A topological Hausdorff space $M$ is called a $k$-dimensional topological manifold if each point $q \in M$ admits a neighborhood $U \subseteq M$ homeomorphic to the open disk $D^k = \{ x \in \mathbb{R}^k \mid ||x|| < 1 \}$ and $M = \bigcup_{i \in \mathbb{N}} U_i$.
- $k$ is called the dimension of the manifold.

manifold

- manifold with boundary
  A topological Hausdorff space $M$ is called a $k$-dimensional topological manifold with boundary if each point $q \in M$ admits a neighborhood $U_i \subseteq M$ homeomorphic either to the open disk $D^k = \{ x \in \mathbb{R}^k \mid ||x|| < 1 \}$ or the open half-space $\mathbb{R}^{k-1} \times \{ y \in \mathbb{R} \mid y \geq 0 \}$ and $M = \bigcup_{i \in \mathbb{N}} U_i$.

smoothness and orientability

- transition functions
  Let $\{(U_i, \varphi_i)\}$ an union of charts on a $k$-dimensional manifold $M$, with $\varphi_i: U_i \to D^k$. the homeomorphisms $\varphi_{ij}: \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ such that $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$ are called transition functions.

smoothness and orientability

- smooth manifold
  A $k$-dimensional topological manifold with (resp. without) boundary is called a smooth manifold with (resp. without) boundary, if all transition functions $\varphi_{ij}$ are smooth.
- orientability
  A manifold $M$ is called orientable if there exists an atlas $\{(U_i, \varphi_i)\}$ on it such that the Jacobian of all transition functions is positive for all intersecting pairs of regions.
Examples

- 3-manifolds with boundary:
  - a solid sphere, a solid torus, a solid knot

- 2-manifolds:
  - a sphere, a torus

- 2-manifold with boundary:
  - a sphere with 3 holes, single-valued functions (scalar fields)

- 1 manifold:
  - a circle, a line

Metric space

A metric space is a set where a notion of distance (called a metric) between elements of the set is defined formally.

\[ (X, d) \]

where

- \( X \) is a set
- \( d \) is a metric on \( X \) (also called distance function), i.e., a function
  \[ d : X \times X \to \mathbb{R} \]
  such that
  - \( d(x, y) \geq 0 \) (non-negative)
  - \( d(x, y) = 0 \) iff \( x = y \) (identity)
  - \( d(x, y) = d(y, x) \) (symmetry)
  - \( d(x, z) \leq d(x, y) + d(y, z) \) (triangle inequality)

Transformations

- Congruence
  - two objects are congruent if one can be transformed into the other by rigid movements (translation, rotation, reflection – not scaling)

- Similarity
  - two geometrical objects are called similar if one can be obtained by the other by uniform stretching. Formally, a similarity of a Euclidean space \( S \) is a function \( f : S \to S \) that multiplies all distances by the same positive scalar \( r \), so that:
    \[ d(f(x), f(y)) = r d(x, y) \quad \forall x, y \in S \]

- Affinity
  - it preserves collinearity, i.e., maps parallel lines into parallel lines and preserve ratios of distances along parallel lines
  - it is equivalent to a linear transformation followed by a translation
Homeo- & Diffeo- morphisms

A homeomorphism between two topological spaces $X$ and $Y$ is a continuous bijection $h: X \rightarrow Y$ with continuous inverse $h^{-1}$.

Given $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, if the smooth function $f: X \rightarrow Y$ is bijective and $f^{-1}$ is also smooth, the function $f$ is a diffeomorphism.

Transformations and Similarities

An isometry is a bijective map between metric spaces that preserves distances:

\[ f: X \rightarrow Y, \quad d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2) \]

Looking for the right metric space...
- the Euclidean distance $d(x, y) = \sum_{i=1}^{n} (x_i - y_i)^2$
- geodesic distances, diffusion distances, ...

Invariance and Isometries

A property invariant under isometries is called an intrinsic property.

Examples:
- the Gaussian curvature $K$
- the first fundamental form
- the geodesic distance
- the Laplace-Beltrami operator

Geodesic Distance

The arc length of a curve $\gamma$ is given by $\int_{\gamma} ds$. Minimal geodesics: shortest path between two points on the surface. Geodesic distance between $P$ and $Q$: length of the shortest path between $P$ and $Q$. Geodesic distances satisfy all the requirements for a metric. A Riemannian surface carries the structure of a metric space whose distance function is the geodesic distance.
Gromov-Hausdorff distance

Let \((X, d_X), (Y, d_Y)\) be two metric spaces and \(C \subset X \times Y\) a correspondence, the distortion of \(C\) is:

\[
\text{dis}(C) = \sup_{(x,y),(x',y') \in C} |d_X(x, x') - d_Y(y, y')|
\]

The Gromov-Hausdorff distance is

\[
d_{\text{GH}}(X, Y) = \frac{1}{2} \inf_C \text{dis}(C)
\]


Properties

The Gromov-Hausdorff distance is parametric with respect to the choice of metrics on the spaces \(X\) and \(Y\).

Common choices

- Euclidean distance (extrinsic geometry)
- Geodesic distance (intrinsic geometry) or, alternatively, diffusion distance

\[
d^2_{X,t}(x,y) = \sum_{i=0}^{\infty} e^{-2\lambda_i t} \langle \psi_i(x) - \psi_i(y) \rangle^2
\]

where \((\lambda_i, \psi_i)\) is the eigensystem of the Laplacian operator and \(t\) is time.

Concepts in Action

- Surface correspondence
- Attribute transfer
- Surface tracking
- Shape analysis (brain imaging)
- Symmetry detection
- Compression
- Completion
- Matching
- Beautification
- Alignment
-...

...stay tuned.... see the Michael Bronstein’s talk

References

- J. Jost, Riemannian geometry and geometric analysis, Universitext, 1979
- M. Gromov, Metric structures for Riemannian and Non-Riemannian spaces, Progress in Mathematics 152, 1999

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